International Journal of Advanced Technology & Engineering Research (IJATER)
International Conference on "Recent Advancement in Science & Technology" (ICRAST 2017)

COMMON FIXED POINT THEOREM FOR (ϕ, ψ)- WEAK CONTRACTIONS IN FUZZY METRIC SPACE USING INTEGRAL TYPE INEQUALITY

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Abstract :- In this paper we prove common fixed point theorem for four mappings in fuzzy metric space and employing the idea of altering distances ,we extend the notion of (ϕ, ψ)- weak contractions in fuzzy metric space using integral type inequality.

Key words:- common fixed point , fuzzy metric space, generalized weak contraction, weakly compatible maps.

Introduction:-

In 1965, L.A.Zadeh [27] introduced the concept of fuzzy set. Kramosil and Michalek [17] introduced the concept of fuzzy metric space which is modified by George and Veeramani [6,7] and they obtained a Hausdorff Topology on the modified class of fuzzy metric space .Also Grabiec [8] first defined the Banach contraction in a fuzzy metric spaces and extended some fixed point theorems of Banach and Edelstein to fuzzy metric space. Sharma[23] extended some results of fixed point theory for compatible maps in fuzzy metric spaces in 2002,At the similar time, Gregory and Sapena [9] introduced the idea of fuzzy contractive mapping and proved fixed point theorems in varied classes of complete fuzzy metric spaces.After that Mihet [18] proposed a fuzzy fixed point theorem for (weak) Banach contraction in *M*-complete fuzzy metric spaces, he further improved the fixed point theory for various contraction mappings in fuzzy metric spaces besides introducing variants of some new contraction mappings. On the other hand, Khan et al. [16] employed the idea of altering distance in metric fixed point results in 1984.
Recently the idea of altering function has been utilized by many authors. The involvement of altering distance sometimes requires special type of techniques as the triangular inequality does not remain directly applicable. Recently fixed point theorems for ψ weak contraction in complete metric space were proved by Rhodes [21].
In this paper, the concept of (ϕ, ψ) weak contractions is introduced in fuzzy metric space and the same is utilized to prove results on existence and uniqueness of fixed points for such maps using integral type inequality.

Preliminaries :-

Definition 1[27]: A fuzzy set is a function A defined on a non empty set X with values in [0,1].

Definition 2 [16]: A function $\phi: [0,\infty) \to [0,\infty)$ is an altering distance function if $\phi(t)$ is monotonically nondecreasing, continuous and $\phi(t) = 0$ if and only if $t = 0$.

Definition 3[22]: A binary operation$*$: [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if it satisfies the following conditions:
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for every $a \in [0,1]$,
(4) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Definition 4[6]: The triplet $(X, M,*)$ is a fuzzy metric space where X is an arbitrary set,$*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0,\infty)$ satisfying the following conditions:
$(i)\ M(x,y,t) > 0,$
$(ii)\ M(x,y,t) = 1\ iff\ x = y,$
$(iii)\ M(x,y,t) = M(y,x,t),$
$(iv)\ M(x,y,t) * M(y,z,s) \leq M(x,z,t + s),$
$(v)\ M(x,y,\cdot) : (0,\infty) \to [0,1]\ is\ continuous,$
$for\ every\ x,y,z \in X\ and\ s,t > 0.$
Here $M(x,y,t)$ is generally interpreted as the measure of nearness between x and y with respect to t.Moreover, it is also well known that $M(x,y,\cdot)$ is non decreasing for all $x,y \epsilon X$.

Definition 5[26]: Let $(X, M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for every $0 < \varepsilon < 1$ and for every $t > 0$, there is $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for every $n,m \geq n_0$.

Definition 6[26]: A sequence $\{x_n\}$ in X is said to be G −Cauchy sequence (i.e.Cauchy sequence in the sense of Grabiec [8]) if $M(x_n, x_{n+k}, t) \to 1$ as $n \to \infty$, for every $k \epsilon$ N and for every $t > 0$.

Definition 7[26]: A fuzzy metric space $(X, M,*)$ is said to be complete (respectively G-complete) if every Cauchy sequence (respectively G-sequence) is convergent.

Remark :-[10] In a metric space (X, d),if we define (with $a * b = ab\ \forall a,b \in [0,1]$) $M_d(x,y,t) = \frac{kt^n}{kt^n + md(x,y)}$ for every $(x,y,t) \in X \times X \times (0,\infty)$ where k, m and n are positive real numbers ,then $(X, M_d,*)$is a fuzzy metric space.Thus, every metric induces a fuzzy metric. The fuzzy metric given by $M_d(x,y,t) = \frac{t}{t+d(x,y)}$ for every $(x,y,t) \in X \times X \times (0,\infty)$ is called standard fuzzy metric.

Definition 8 [26]: Let $(X, M,*)$ be a fuzzy metric space and $p, A : X \to X$ be two maps. A point x in X is called coincidence point (common fixed point) of p and A if $px = Ax$ ($px = Ax = x$). The maps $p, A : X \to X$ are weakly Compatible if they commute on the set of their coincidence points.

Definition 9 [26]: Let $(X, M,*)$ be a fuzzy metric space and $p : X \to X$ be a map. The map $A : X \to X$ is called a ψ-weak contraction with respect to p if there exists a function $\psi : [0, \infty) \to [0, \infty)$ With $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ such that

$$\frac{1}{M(Ax, Ay, t)} - 1 \leq \left(\frac{1}{M(px, py, t)} - 1\right) - \psi\left(\frac{1}{M(px, py, t)} - 1\right) \quad \ldots\ldots (1)$$

holds for every $x, y \in X$ and each $t > 0$.If the map p is the identity map, then the map $A : X \to X$ is called a ψ -weak contraction.

Applying the notion of altering distance function, we adopt the notion of (ϕ, ψ)- weak contraction in fuzzy metric spaces.

Definition 10 [26]: Let $(X, M,*)$ be a fuzzy metric space and $p : X \to X$ be a map. The map $A : X \to X$ is called a (ϕ, ψ)- weak contraction with respect to p if there exist a function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function ϕ such that

$$\phi\left(\frac{1}{M(Ax, Ay, t)} - 1\right) \leq \phi\left(\frac{1}{M(px, py, t)} - 1\right) - \psi\left(\frac{1}{M(px, py, t)} - 1\right) \quad \ldots\ldots\ldots\ldots(2)$$

holds for every $x, y \in X$ and each $t > 0$. If the map p is the identity map, then the map $A : X \to X$ is called a (ϕ, ψ)- weak contraction. Suppose

$$m(p, q, A, B) = min\{M(qx, py, t), M(Ax, qx, t), M(By, py, t)\}.$$

Definition 11 [26]: Let $(X, M,*)$ be a fuzzy metric space and $p, q, A, B : X \to X$ be four maps. The pair $\{A, B\}$ is called a generalized (ϕ, ψ)- weak contraction of integral type with respect to $\{p, q\}$ if there exist a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function ϕ satisfying, for every $x, y \in X$ and each $t > 0$ the condition

$$\int_0^{\phi\left(\frac{1}{M(Ax,By,t)}-1\right)} \varphi(s)ds \leq \int_0^{\phi\left(\frac{1}{m(p,q,A,B)}-1\right)} \varphi(s)ds - \int_0^{\psi\left(\frac{1}{m(p,q,A,B)}-1\right)} \varphi(s)ds \quad \ldots\ldots\ldots\ldots (3)$$

Where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable map which is summable on each compact subset of $[0,\infty)$ and such that for all $\in > 0$,

$\int_0^{\epsilon} \varphi(s)ds > 0$

If $B = A$ and $p = q$, then the map $A : X \to X$ is called a generalized (ϕ, ψ)- weak contraction of integral type with respect to p.

Main Result :

Theorem 1. Let $(X, M,*)$be a fuzzy metric space and $p, q, A, B: X \to X$ be four maps such that the pair {A,B} is a generalized (ϕ, ψ)- weak contraction of integral type with respect to $\{p, q\}$.If $A(X) \subseteq p(X)$,$B(X) \subseteq q(X)$,one of $p(X), q(X), B(X)$ and $A(X)$ is a G-complete subspace of X, then the pairs $\{p, B\}$ and $\{q, A\}$ have a coincidence point each provided ψ is continuous. Moreover, p, q, B and A have a unique common fixed point in X provided the pairs $\{p, B\}$ and $\{q, A\}$ are weakly compatible.

Proof. Let x_0 be an arbitrary element in X .Since $B(X) \subseteq q(X)$ and $A(X) \subseteq p(X)$.
For every $n \geq 0$ we define the sequence $\{y_n\} \subset X$ by $y_{2n} = Ax_{2n} = px_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = qx_{2n+2}$.
We assume that $y_{2n} = y_{2n+1}$ for some n.
Then by (3) we have $y_{2n+1} = y_{2n+2}$ and so $y_m = y_{2n}$ $\forall m > 2n$.
Therefore the sequence $\{y_n\}$ is Cauchy. The similar conclusion holds if $y_{2n+1} = y_{2n+2}$ for some n.
Suppose $y_n \neq y_{n+1}, \forall n$
Then for $x = x_{2n}$ and $y = x_{2n-1}$ we have

$$m(p, q, A, B) = \min\{M(qx_{2n}, px_{2n-1}, t), M(Ax_{2n}, qx_{2n}, t), M(Bx_{2n-1}, px_{2n-1}, t)\}$$

$$=\min\{M(y_{2n-1}, y_{2n-2}, t), M(y_{2n}, y_{2n-1}, t), M(y_{2n-1}, y_{2n-2}, t)\}$$

Therefore if $m(p, q, A, B) = M(y_{2n}, y_{2n-1}, t)$we obtain

$$\int_0^{\phi\left(\frac{1}{M(y_{2n},y_{2n-1},t)}-1\right)} \varphi(s)ds \leq \int_0^{\phi\left(\frac{1}{M(y_{2n},y_{2n-1},t)}-1\right)} \varphi(s)ds - \int_0^{\psi\left(\frac{1}{M(y_{2n},y_{2n-1},t)}-1\right)} \varphi(s)ds$$

Which implies that $\int_0^{M(y_{2n},y_{2n-1},t)} \varphi(s)ds$=1, which is a contradiction as $y_n \neq y_{n+1}$ $\forall$ n.
Then we must have $m(p, q, A, B) = M(y_{2n-1}, y_{2n-2}, t)$ and hence

$$\int_0^{\phi\left(\frac{1}{M(y_{2n},y_{2n-1},t)}-1\right)} \varphi(s)ds \leq \int_0^{\phi\left(\frac{1}{M(y_{2n-1},y_{2n-2},t)}-1\right)} \varphi(s)ds - \int_0^{\psi\left(\frac{1}{M(y_{2n-1},y_{2n-2},t)}-1\right)} \varphi(s)ds$$

$$< \int_0^{\phi\left(\frac{1}{M(y_{2n-1},y_{2n-2},t)}-1\right)} \varphi(s)ds.$$

By using the previous concept we obtain the same inequality for $x = x_{2n-2}$ and $y = x_{2n-1}$.Consequently if we consider the fact that the function ϕ is non decreasing we have

$\int_0^{M(y_n,y_{n+1},t)} \varphi(s)ds > \int_0^{M(y_{n-1},y_n,t)} \varphi(s)ds, \forall\, n$ and hence $\{M(y_{n-1}, y_n, t)\}$ is an increasing sequence of positive real numbers in $(0,1]$.
Let $S(t) = \int_0^{M(y_{n-1},y_n,t)} \varphi(s)ds$ as $n \to \infty$.
Then we show that $S(t) = 1\ \forall\, t > 0$.If not then there exist $t > 0$ such that $S(t) < 1$ .Then on taking $n \to \infty$ in above inequality, we obtain

$$\int_0^{\phi\left(\frac{1}{S(t)}-1\right)} \varphi(s)ds \leq \int_0^{\phi\left(\frac{1}{S(t)}-1\right)} \varphi(s)ds - \int_0^{\psi\left(\frac{1}{S(t)}-1\right)} \varphi(s)ds$$

a contradiction .Therefore $\int_0^{M(y_n,y_{n+1},t)} \varphi(s)ds \to 1$ as $n \to \infty$.
Now for each positive integer p

$$\int_0^{M(y_n,y_{n+p},t)} \varphi(s)ds \geq \int_0^{M\left(y_n,y_{n+1},\frac{t}{p}\right)*M\left(y_{n+1},y_{n+2},\frac{t}{p}\right)*\dots\dots*M\left(y_{n+p-1},y_{n+p},\frac{t}{p}\right)} \varphi(s)ds$$

As $n \to \infty$ it follows that

$$\int_0^{M(y_n,y_{n+p},t)} \varphi(s)ds \geq 1 * 1 * 1 * 1 \dots\dots\dots\dots\dots .1 = 1$$

Hence $\{y_n\}$ is a G-Cauchy sequence.
If $q(X)$ is G-complete ,then there exist $z \in q(X)$ such that $y_n \to z$ as $n \to \infty$.
Clearly $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} px_{2n+1} = z$ and $\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} qx_{2n+2} = z$.
Let $u \in X$ be such that $qu = z$. We show that u is a coincidence point of A and q .Assume $Au \neq qu$.Now for $x = u$ and $y = x_{2n+1}$ we have

$$m(p, q, A, B) = \min\{M(qu, px_{2n+1}, t), M(Au, qu, t), M(Bx_{2n+1}, px_{2n+1}, t)\}$$

and

$$\int_0^{\phi\left(\frac{1}{M(Au,Bx_{2n+1},t)}-1\right)} \varphi(s)ds \leq \int_0^{\phi\left(\frac{1}{m(p,q,A,B)}-1\right)} \varphi(s)ds - \int_0^{\psi\left(\frac{1}{m(p,q,A,B)}-1\right)} \varphi(s)ds$$

for every $t > 0$ which on taking $n \to \infty$ gives rise

$$\int_0^{\phi\left(\frac{1}{M(Au,z,t)}-1\right)} \varphi(s)ds < \int_0^{\phi\left(\frac{1}{M(Au,z,t)}-1\right)} \varphi(s)ds$$

Which is contradiction yielding thereby Au=z.
Therefore z is a point of coincidence for the pair $\{q, A\}$.
Since the pair $\{q, A\}$ is weakly compatible, we have $qz = qAu = Aqu = Az$.
If not then for $x = z$ and $y = x_{2n+1}$ we have $\forall\, t > 0$.

$$\mathrm{m(p,q,A,B)} = \min\{M(qz, px_{2n+1}, t), M(Az, qz, t), M(Bx_{2n+1}, px_{2n+1}, t)\} \to M(Az, z, t)$$

Now by

$$\int_0^{\phi\left(\frac{1}{M(Az,Bx_{2n+1},t)}-1\right)} \varphi(s)ds \le \int_0^{\phi\left(\frac{1}{m(p,q,A,B)}-1\right)} \varphi(s)ds - \int_0^{\psi\left(\frac{1}{m(p,q,A,B)}-1\right)} \varphi(s)ds$$

Which on taking $n \to \infty$ reduces to

$$\int_0^{\phi\left(\frac{1}{M(Az,z,t)}-1\right)} \varphi(s)ds < \int_0^{\phi\left(\frac{1}{M(Az,z,t)}-1\right)} \varphi(s)ds$$

a contradiction yielding therefore there by $Az = z$ .To show that z is also fixed point of the pair $\{p, B\}$,it is observe that $A(X) \subseteq p(X)$then there is some element $v \in X$ such that $Az = pv$.Thus $Az = pv = qz = z$.We claim that $Bv = z$.If it is not then by using (3) with $x = z$ and $y = v$.We again get a contradiction implying thereby $Bv = z$. So $Bv = pv = z$ which shows that v is a coincidence point of B and p.By weak compatibility of the pair $\{p, B\}$, we obtain $Bv = Bpv = ppv = pz$.

Finally arguing via contradiction by using (3) we get $Sz = z$ and hence $Az = Bz = pz = qz = z$. Clearly proceeding on the previous lines .One can obtain the same conclusion in case (instead of $q(X)$) one of $p(X), B(X)$ and $A(X)$ is a G-complete subset of X. The uniqueness of the common fixed point z is also an easy consequence of condition (3) so we skip the details .This completes the proof.

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