International Journal of Advanced Technology & Engineering Research (IJATER)
International Conference on "Recent Advancement in Science & Technology" (ICRAST 2017)

A COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACE USING OCCASIONALLY WEAK COMPATIBILITY

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Abstract

In this paper we establish common fixed point theorem for six mapping in fuzzy metric space using the notation of occasionally weak compatibility.

Keywords: Fuzzy Metric Space, Fixed Point Theorem, Occasionally Weakly Compatible.

1 INTRODUCTION

The concept of a fuzzy set was first introduced by Zadeh L.A. [12] and developed a basic frame work to treat mathematically the fuzzy phenomena or systems which due to in trinsic indefiniteness, cannot themselves be characterized precisely. Fuzzy metric spaces have been introduced by Kramosil and Michalek [8] and George and Veersamani [4] modified the notion of fuzzy metric with help of continuous t-norms. Singh and Chauhan [10] introduced the concept of compatible mappings of Fuzzy metric space and proved the common fixed point theorem. Jungck et.al. [7] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Using the concept of compatible maps of type (A), Jain et.al. [6] proved a fixed point theorem for six self-maps in a fuzzy metric space.
Various authors have discussed and studied extensively various results on coincidence, existence and uniqueness of fixed and common fixed points by using the concept of weak commutativity, compatibility, non-compatibility and weak compatibility for single and set valued maps satisfying certain contractive conditions in different spaces and they have been applied to diverse problems. Al-Thagafi and Shahzad [1] weakened the concept of compatibility by giving a new notion of occasionally weakly compatible (owc) maps which is more general among the commutativity concepts. Most recently, Bouhadjera and Thobie [2], weakened the concept of occasionally weak compatibility and reciprocal continuity in the form of sub compatibility and sub sequential continuity respectively and proved some interesting results with these concepts in metric space. We prove common fixed point theorem for six mappings using the concept of occasionally weakly compatible.

2.1 DEFINITIONS AND PRELIMINARIES:

2.1. Definition: A binary operation$*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0, 1]$,

(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Example of t -norm are $a * b = a\, b$ and $a * b = \min\{a, b\}$

2.2. Definition: A 3-tuple (X, M, $*$) is said to be fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is fuzzy sets in $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$.

(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(v) $M(x, y, \cdot) : [0,\infty) \to [0, 1]$ is continuous,
(vi) $\lim_{n\to\infty} M(x, y, t) = 1$.

Note that $M(x,y,t)$ can be considered as the degree of nearness between x and y with respect to t. We define $x = y$ with $M(x,y,t) = 1$ for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space induces a fuzzy metric space.

2.3. Example: Let (X,d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and all $t > 0$. Then (X, M, $*$) is a fuzzy metric space. It is called the fuzzy metric space introduced by d.

2.4. Definition: A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, $*$) is called Cauchy sequence if and only if for each $\varepsilon > 0$, $t > 0$, there exits $n_0 \in N$ such that $M(x_n, x_m, t) > 1- \varepsilon$ for all $n, m \ge n_0$.
The sequence $\{x_n\}$ is said to converges to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exist $n_0 \in N$ such that $M(x_n, x, t) > 1- \varepsilon$ for all $n \ge n_0$.

2.5. Definition: A Fuzzy metric space (X, M, $*$) is said to be complete if every Cauchy sequence in it converges to a point in it.

2.6. Definition: Let S and T be self-mapping of fuzzy metric space (X, M, $*$) are said to be compatible if and only $M(STx_n, TSx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Tx_n , $Sx_n \to p$ for some p in X as $n \to \infty$.

2.7. Definition: Let S and T be self-mapping of a Fuzzy metric space (X, M, $*$) are said to be weakly compatible if they commute at their coincidence points, i.e. $Sx = Tx$ for some $x \in X$ then $STx = TSx$.

2.8. Definition: Let S and T be self-mapping of a Fuzzy metric space (X, M, $*$) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of S and T at which S and T commute.

2.1. Lemma: Let $\{u_n\}$ is a sequence in a fuzzy metric space (X, M, $*$). If there exist $k \in (0, 1)$ such that

$M(u_{n+2}, u_{n+1}, kt) \ge M(u_{n+1}, u_n, t)$ for all $t > 0$ $n \in N$.

Then $\{u_n\}$ is Cauchy sequence in X.

2.2. Lemma: Let (X, M, $*$) be a fuzzy metric space. If there exist $k \in (0, 1)$ such that for all $x, y \in$ X
M$(x, y, kt) \geq$ M(x, y, t) for all $t > 0$ then $x = y$.

2.3. Lemma: The only t-norm $*$ satisfying $r * r \geq r$ for all r $\in [0, 1]$ is the minimum t- norm, that is $a * b$ = min $\{ a, b\}$ for all $a, b \in [0, 1]$.

2.1. Theorem [9]: Let A, B, P and Q be self –mappings of a complete fuzzy metric space (X, M, $*$) satisfying the following:

(i) For any x, y in X, and for all $t > 0$ there exists k $\in$ (0, 1) such that

$$M\,(Px, Qy, kt) \geq \max\left\{\begin{matrix} M(Ax,\ By,\ t), \\ \frac{1}{2}\begin{pmatrix} M(Px,\ Ax,\ t) \\ +M(Qx,\ Bx,\ t)\end{pmatrix}\end{matrix}\right\}$$

(ii) P(X) $\subseteq$ B(X) and Q(X) $\subseteq$ A(X).

(iii) if one of P(X), B(X), Q(X) and A(X) is complete subset of X then.

(a) P and A have a coincidence point

(b) Q and B have a coincidence point.

If the pair (P, A) and (Q, B) are weakly compatible then A, B, P and Q have a unique common fixed point in X.

MAIN THEOREM

3.1 Theorem: Let (X, M, $*$) be a complete fuzzy metric space with $t * t \geq t$ for all t $\in [0, 1]$. Let A, B, S, T, P and Q be six self –mappings satisfying the following condition:

(a) P(X) $\subset$ ST(X) and Q(X) $\subset$ AB(X);

(b) AB=BA, QB=BQ, ST=TS and PT=TP;

(c) Pair (P, ST) and (Q, AB) are occasionally weakly compatible;

(d) There exist a number $k \in$ (0, 1) such that

$$M\,(Px, Qy, kt) \geq \max\left\{\begin{matrix} M(STx,\ ABy,\ t), \\ \frac{1}{3}\begin{pmatrix} M(Px,\ STx,\ t) + M(Px,\ ABx,\ t) \\ +[M(Px,\ ABy,\ t) * M(ABy,\ STx,\ t)]\end{pmatrix}\end{matrix}\right\}$$

for all $x, y \in$ X

If the range of subspaces P(X) or ST (X) or Q(X) or AB(X) is complete, then A, B, S, T, P and Q have a unique common fixed point.

Proof: As P(X) $\subset$ST(X) and Q(X) $\subset$ AB(X), so we can define sequences $\{x_n\}$ and $\{y_n\}$ in X such that

y_{2n+1}= Px_{2n} = STx_{2n+1} and y_{2n+2} = Qx_{2n+1} = ABx_{2n+2}

By (d), we have

$$M\,(Px_{2n},\ Qx_{2n+1},\ kt) \geq \max\left\{\begin{matrix} M(STx_{2n},\ ABx_{2n+1},\ t), \\ \frac{1}{3}\begin{pmatrix} M(Px_{2n},\ STx_{2n},\ t) + M(Px_{2n},\ ABx_{2n},\ t) \\ +[M(Px_{2n},\ ABx_{2n+1},\ t) * M(ABx_{2n+1},\ STx_{2n},\ t)]\end{pmatrix}\end{matrix}\right\}$$

$$M\ (y_{2n+1}, y_{2n+2}, kt) \geq \max\left\{\begin{matrix} M(y_{2n},\ y_{2n+1},\ t), \\ \frac{1}{3}\begin{pmatrix} M(y_{2n+1},\ y_{2n},\ t) + M(y_{2n+1},\ y_{2n},\ t) \\ +[M(y_{2n+1},\ y_{2n+1},\ t) * M(y_{2n+1},\ y_{2n},\ t)] \end{pmatrix} \end{matrix}\right\}$$

$$M\ (y_{2n+1}, y_{2n+2}, kt) \geq \max\left\{\begin{matrix} M(y_{2n},\ y_{2n+1},\ t), \\ \frac{1}{3}\begin{pmatrix} M(y_{2n+1},\ y_{2n},\ t) + M(y_{2n+1},\ y_{2n},\ t) \\ +M(y_{2n+1},\ y_{2n},\ t) \end{pmatrix} \end{matrix}\right\}$$

$$M\ (y_{2n+1}, y_{2n+2}, kt) \geq \max\left\{\begin{matrix} M(y_{2n},\ y_{2n+1},\ t), \\ M(y_{2n+1},\ y_{2n},\ t) \end{matrix}\right\}$$

$$M\ (y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n},\ y_{2n+1},\ t)$$

Similarly

$$M\ (y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1},\ y_{2n+2},\ t)$$

There in general

$$M\ (y_n, y_{n+1}, kt) \geq M(y_{n-1},\ y_n,\ t)$$

Hence, by lemma 2.1 $\{y_n\}$ is Cauchy sequence in X. By completeness of X, $\{y_n\}$ converges some point z in X. Therefore sub sequences$\{y_{2n}\}$, $\{y_{2n+1}\}$, $\{y_{2n+2}\}$ converges to point z .

i.e.

$$\lim_{n\to\infty} \mathrm{P}x_{2n} = \lim_{n\to\infty} \mathrm{ST}x_{2n+1} = \lim_{n\to\infty} \mathrm{Q}x_{2n+1} = \lim_{n\to\infty} \mathrm{AB}x_{2n+2} = z$$

Since Q(X) ⊆ AB(X), there exist a point $u \in$ X such that AB$u = z$

Putting $x = x_{2n}$ and $y = u$, then by (d), we have

$$M\ (Px_{2n},\ Qu, kt) \geq \max\left\{\begin{matrix} M(STx_{2n},\ ABu\ \ t), \\ \frac{1}{3}\begin{pmatrix} M(Px_{2n},\ STx_{2n},\ t) + M(Px_{2n},\ \ ABx_{2n},\ t) \\ +[M(Px_{2n},\ ABu,\ t) * M(ABu,\ STx_{2n},\ t)] \end{pmatrix} \end{matrix}\right\}$$

$$M\ (z,\ Qu,\ kt) \geq \max\left\{\begin{matrix} M(z,\ z,\ t), \\ \frac{1}{3}\begin{pmatrix} M(z,\ z,t) + M(\ z,z,t) \\ +[M(z,\ z,\ t) * M(z,\ z,\ t)] \end{pmatrix} \end{matrix}\right\}$$

$$M(Qu, z,\ kt) \geq \max\left\{\begin{matrix} 1, \\ \frac{1}{3}(1+1+1) \end{matrix}\right\}$$

$$M\ (Qu,\ z,\ kt) \geq \max\{1, 1\}$$

$M\ (Qu, z,\ kt) \geq 1$

$Qu = u$

Therefore $ABu = Qu = z$

Since P(X) ⊆ ST(X), there exist a point $u \in$ X such that $STu = z$

Putting $x = u$ and $y = x_{2n+1}$, then by (d), we have

$$M\ (Pu,\ Qx_{2n+1}, kt) \geq \max \begin{Bmatrix} M(STu,\ ABx_{2n+1}\ \ t), \\ \frac{1}{3}\begin{pmatrix} M(Pu,\ STu,\ t) + M(Pu,\ \ ABu,\ t) \\ +[M(Pu,\ ABx_{2n+1},\ t) * M(ABx_{2n+1},\ STu,\ t)] \end{pmatrix} \end{Bmatrix}$$

$$M\ (Pu,\ z,\ kt) \geq \max \begin{Bmatrix} M(z,\ z,\ t), \\ \frac{1}{3}\begin{pmatrix} M(Pu,\ z,t) + M(\ Pu,z,t) \\ +[M(Pu,\ z,\ t) * M(z,\ z,\ t)] \end{pmatrix} \end{Bmatrix}$$

$$M\ (Pu,\ z,\ kt) \geq \max \begin{Bmatrix} M(z,\ z,\ t), \\ \frac{1}{3}\begin{pmatrix} M(Pu,\ z,t) + M(\ Pu,z,t) \\ +M(Pu,\ z,\ t) \end{pmatrix} \end{Bmatrix}$$

$$M(Pu, z,\ kt) \geq \max \begin{Bmatrix} 1, \\ M(Pu,\ z,\ t) \end{Bmatrix}$$

$M\ (Pu, z,\ kt) \geq 1$

$Pu = z$

Therefore $ABu = Qu = Pu = STu = z$

Since the pair (Q, AB) is occasionally weakly compatible, we have
$ABu = Qu$ Implies that $QABu = ABQu$ i.e. $Qz = ABz$
Now we show that z is a fixed point of Q.
Put $x =\ u$ and $y = z$ in inequality (d), we get

$$M\ (Pu,\ Qz, kt) \geq \max \begin{Bmatrix} M(STu,\ ABz,\ t), \\ \frac{1}{3}\begin{pmatrix} M(Pu,\ STu,\ t) + M(Pu,\ \ ABu,\ t) \\ +[M(Pu,\ ABz,\ t) * M(ABz,\ STu,\ t)] \end{pmatrix} \end{Bmatrix}$$

$$M\ (z,\ Qz,\ kt) \geq \max \begin{Bmatrix} M(z,\ Qz,\ t), \\ \frac{1}{3}\begin{pmatrix} M(z,\ z,t) + M(\ z,z,t) \\ +[M(z,\ Qz,\ t) * M(Qz,\ z,\ t)] \end{pmatrix} \end{Bmatrix}$$

$$M\ (z,\ Qz,\ kt) \geq \max \begin{Bmatrix} M(z,\ Qz,\ t), \\ \frac{1}{3}\begin{pmatrix} M(z,\ z,t) + M(\ z,z,t) \\ +M(z,\ z,\ t) \end{pmatrix} \end{Bmatrix}$$

$$M(Qu, z,\ kt) \geq \max \begin{Bmatrix} M(z,\ Qz,\ t), \\ \frac{1}{3}(1+1+1) \end{Bmatrix}$$

$$M\ (Qz,\ z,\ kt) \geq \max\ \{M(z,\ Qz,\ t), 1\}$$

$$M\ (Qz, z,\ kt) \geq 1$$

$Qz = z$

Therefore $ABz = Qz = z$

Similarly, pair of map (P, ST) is occasionally weakly compatible, we have
$Pz = STz = z$
Now we show that $Bz = z$, by putting $x = x_{2n+1}$ and $y = Bz$ in inequality (d), we get

$$M\ (Px_{2n+1},\ QBz, kt) \geq \max \begin{Bmatrix} M(STx_{2n+1},\ AB(Bz)\ t), \\ \frac{1}{3}\begin{pmatrix} M(Px_{2n+1},\ STx_{2n+1},\ t) + M(Px_{2n+1},\ ABx_{2+1n},\ t) \\ +[M(Px_{2n+1},\ AB(Bz),\ t) * M(AB(Bz),\ STx_{2n+1},\ t)] \end{pmatrix} \end{Bmatrix}$$

Since $Qz = ABz = z$
$BQz = B(ABz) = Bz$
Since QB =BQ and AB = BA
So $QBz = AB(Bz) = Bz$
Proceeding limit as n $\rightarrow \infty$ then we get

$$M\ (z,\ Bz,\ kt) \geq \max \begin{Bmatrix} M(z,\ Bz,\ t), \\ \frac{1}{3}\begin{pmatrix} M(z,\ z, t) + M(\ z, z, t) \\ +[M(z,\ Bz,\ t) * M(Bz,\ z,\ t)] \end{pmatrix} \end{Bmatrix}$$

$$M(Bz, z,\ kt) \geq \max \begin{Bmatrix} M(z,\ Bz,\ t), \\ \frac{1}{3}(1+1+1) \end{Bmatrix}$$

$$M\ (Bz,\ z,\ kt) \geq \max\ \{M(z,\ Bz,\ t), 1\}$$

$$M\ (Bz, z,\ kt) \geq 1$$

$Bz = z$

Since $ABz = z$
Therefore $Pz = ABz = Bz = z = Qz = STz$
Finally we show that $Tz = z$ by putting $x = Tz$ and $y = z$ in inequality (d), we get

$$M\ (PTz,\ Qz,\ kt) \geq \max \begin{Bmatrix} M(ST(Tz),\ ABz,\ t), \\ \frac{1}{3}\begin{pmatrix} M(PTz,\ ST(Tz), t) + M(\ PTz, AB(Tz), t) \\ +[M(PTz,\ ABz,\ t) * M(ABz,\ ST(Tz),\ t)] \end{pmatrix} \end{Bmatrix}$$

Since $Pz = STz = z$
$TPz = T(STz) = Tz$
Since PT =TP and ST = TS

So $PTz = ST(Tz) = BTz$

$$M(Tz,\ z,\ kt) \geq \max\left\{\begin{matrix} M(Tz,\ z,\ t), \\ \frac{1}{3}\begin{pmatrix} M(Tz, Tz, t) + M(Tz, Tz, t) \\ +[M(Tz,\ z,\ t) * M(z,\ Tz,\ t)] \end{pmatrix} \end{matrix}\right\}$$

$$M(Tz, z,\ kt) \geq \max\left\{\begin{matrix} M(Tz,\ z,\ t), \\ \frac{1}{3}(1+1+1) \end{matrix}\right\}$$

$$M(Tz,\ z,\ kt) \geq \max\{M(z,\ Tz,\ t), 1\}$$

$$M(Tz, z,\ kt) \geq 1$$

$Tz = z$

Therefore $ABz = Bz = STz = Tz = Pz = Qz = z$

Uniqueness follows easily.

This completes the proof of theorem.

If we put B = T = I, the identity map on X, in Theorem 3.1, we get

Corollary 3.1.1

Let (X, M, $*$) be a complete fuzzy metric space with $t * t \geq t$ for all t $\in$ $[0,\ 1]$. Let A, S, P and Q be four self – mappings satisfying the following condition:

(a) P(X) ⊂ S(X) and Q(X) ⊂ A(X)

(b) Pair (P, S) and (Q, T) are occasionally weakly compatible;

(c) There exists a number k∈ (0, 1) such that

(d) $$M(Px, Qy, kt) \geq \max\left\{\begin{matrix} M(Sx,\ Ay,\ t), \\ \frac{1}{3}\begin{pmatrix} M(Px,\ Sx,\ t) + M(Px,\ Ax,\ t) \\ +[M(Px,\ Ay,\ t) * M(Ay,\ Sx,\ t)] \end{pmatrix} \end{matrix}\right\}$$

for all $x, y \in$ X

If the range of one subspaces is complete, then A, S, P and Q have a unique common fixed point.

If we put A = B = S = T = I in Theorem 3.1, we get

Corollary 3.1.2

Let (X, M, $*$) be a complete fuzzy metric space with $t * t \geq t$ for all t $\in$ $[0,\ 1]$. Let P and Q be occasionally weakly compatible mapping from X into itself. If there exists a number k∈ (0, 1) such that

$$M(Px, Qy, kt) \geq \max\left\{\begin{matrix} M(x,\ y,\ t), \\ \frac{1}{3}\begin{pmatrix} M(Px,\ x,\ t) + M(Px,\ x,\ t) \\ +[M(Px,\ y,\ t) * M(y,\ x,\ t)] \end{pmatrix} \end{matrix}\right\}$$

for all $x, y \in$ X

If the range of one subspaces is complete, then P and Q have a unique common fixed point.

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