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FIXED POINT RESULTS IN METRIC-LIKE SPACES VIA SIMULATION FUNCTIONS AND APPLICATION TO DYNAMIC PROGRAMMING

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Abstract

The aim of this paper is to present some fixed point results in the setting of a metric-like space using simulation functions via admissible mapping. Suitable examples are given to illustrate the usability of the obtained results. As an application existence of common solution for system of functional equations arising in dynamic programming has been studied.

Keywords: metric-like space, common fixed point, α-orbital admissible, Z–contraction, dynamic programming.

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Introduction and Preliminaries

Fixed point theory is a very useful tool for several areas of mathematical analysis and its applications. The notion of metric-like (dislocated) metric spaces was introduced by Hitzler and Seda [7] in 2000 as a generalization of a metric space. They generalized the Banach Contraction Principle [4] in such spaces. Metric-like spaces were discovered by Amini-Harandi [2] who established some fixed point results.

In this paper, we introduce the existence and uniqueness of fixed points of certain mappings via simulation functions in the context of metric-like space. We shall also indicate that several results in the literature can be derived from our main result. Suitable examples and an application existence of common solution for system of functional equations arising in dynamic programming have been studied.

Our results are generalization and extension of result of Khojasteh [4] fixed points of certain mappings via simulation functions in the context of metric-like space. We shall also indicate that several results in the literature can be derived from our main result. Numerical examples are illustrated and an immediate application of our investigation towards the existence of common solution for system of functional equations arising in dynamic programming has been studied.

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}^*$ will denote the set of real numbers, the set of nonnegative real numbers and the set of positive integer numbers respectively.

Definition 1.1. (see [2]) Let X be a nonempty set, a mapping $\sigma: X \times X \to \mathbb{R}^+$ if for all $x, y, z \in X$, such that the following conditions holds

(σ_1) $\sigma(x, y) = 0$ implies $x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, y) \le \sigma(x, z) + \sigma(z, y)$.

Then (X, σ) is said to be a metric-like space.

Note that, a metric-like satisfies all the conditions of metric except that σ(x, x) may be positive for $x \in X$. Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls $B_\sigma(x, \varepsilon) = \{y \in X: | \sigma(x, y) - \sigma(x, x) | < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2. (see [2]) Let (X,σ) is a metric-like space and let $\{x_n\}$ be any sequence in X and $x \in X$. Then

(1) $\{x_n\}$ is converges to a point x w.r.t. τ_σ, if and only if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$.

(2) A function $f : (X, \sigma) \to (X, \sigma)$ is continuous if for any sequence $\{x_n\}$ in X such that $\sigma(x_n, x) \to \sigma(x, x)$ as $n \to \infty$, we have $\sigma(fx_n, fx) \to \sigma(fx, fx)$ as $n \to \infty$,

Note that every partial metric space is a metric-like space, but the converse may not be true.

Example 1.1. Take X = {1, 2, 3} and consider the metric-like $\sigma: X \times X \to \mathbb{R}_0^+$ given by

$$\sigma(1, 1) = 0, \quad \sigma(2, 2) = \tfrac{3}{2}, \quad \sigma(3, 3) = \tfrac{5}{6},$$
$$\sigma(2,1) = \sigma(1,2) = \tfrac{3}{5},$$
$$\sigma(3,1) = \sigma(1,3) = \tfrac{4}{5}, \ \sigma(2,3) = \sigma(3,2) = \tfrac{7}{10}.$$

Since $\sigma(2; 2) \neq 0$; so σ is not a metric and since $\sigma(2; 2) > \sigma(2; 1)$; so σ is not a partial metric.

Next, we give the concepts of σ -convergence, σ -Cauchy sequence, σ -continuity, σ -closed and σ -complete in metric-like space.

Definition 1.3. (see [8]) Let (X, σ) is a metric-like space. Then a sequence $\{x_n\}$ in X is called

(1) σ-convergent if there exists $x \in X$ such that $\lim_{n\to\infty} \sigma(x_n, x) = 0$.

(2) $\{x_n\}$ is said to be σ-Cauchy if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite.

(3) A subset C of (X, σ) is called σ-closed if the σ-limit of a σ-convergent sequence of C is still in C.

(4) A subset C of (X, σ) is called σ-complete if for each σ-Cauchy in C is σ-convergent and its σ-limit belongs to C.

(5) A function $f : X \to X$ is σ-continuous when if
$$\sigma(x_n, x) \to x \text{ then } \sigma(fx_n, fx) \to 0 \text{ as } n \to \infty.$$

The concept of α-admissibility was first introduced by Samet *et al.* [12] Later, Very recently Popescu [11] proposed the concept of triangular α-orbital admissible as a refinement of the triangular α -admissible notion, defined in [9].

Definition 1.4. (see [12]) For a nonempty set X, let $\alpha: X \times X \to [0,\infty)$ and $f : X \to X$ are mappings. Then

(1) the mapping f is α-orbital admissible mapping if
$$\alpha(x, fy) \geq 1 \Rightarrow \alpha(fx, f^2y) \geq 1, \forall x, y \in X.$$

(2) the mapping f is called triangular α-orbital admissible if $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1, \forall x,y,z \in X$.

Recently, Khojasteh *et al.* [10] introduced a new class of mappings called simulation functions. Using the above concept, they proved several fixed point theorems and showed that many known results in literature are simple consequences of their obtained results. Later, Argoubi *et al.* [1] slightly modified the definition of simulation functions by withdrawing a condition. Let Z^* be the set of simulation functions in the sense of Argoubi *et al.* [1].

Definition 1.6. (see [10]) A simulation function is a mapping $\zeta: [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(ζ_2) if $\{x_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $lim_{n\to\infty} s_n = l \in (0,\infty)$, then $limsup_{n\to\infty}\zeta(t_n, s_n) < 0$.

Example 1.2. (see [10]) Let $\zeta_\lambda: [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by
$$\zeta_\lambda(t,s) = \begin{cases} 1, & if\ (t,s) = (0,0) \\ \lambda s - t, & otherwise \end{cases}$$
Where $\lambda \in (0,1)$. Then $\zeta_\lambda \in Z^*$.

Main Results

Definition 1.1. Let $f: C \to C$ be a mapping defined on a metric space (X,σ). If there exist simulation function $\zeta \in Z^*$ and $\alpha: C \times C \to [0,\infty)$, such that
$$\zeta(\alpha(x,y)\sigma(fx,fy), M(x,y)) \geq 0 \qquad (2.1)$$
for all $x, y \in C$, where
$$M(x,y) = max\left\{\begin{array}{c} \sigma(x,y), \sigma(x,fx), \sigma(y,fy), \\ \frac{\sigma(x,fy)+\sigma(fx,y)}{2}, \frac{[1+\sigma(y,fy)]\sigma(x,fx)}{1+\sigma(fx,fy)} \end{array}\right\} \qquad (2.2)$$
then we say that f is an α -orbital admissible Z-contraction with respect to ζ.

Remark 2.1. If α(x, y) = 1, then f turns into a Z-contraction with respect to ζ.

Theorem 2.1. Let C be a σ-closed subset of a metric-like space (X,σ) where Let $f: C \to C$ be an α –orbital admissible Z-contraction with respect to ζ. Suppose that the following conditions hold:

(S_1) f is triangular α-orbital admissible;

(S_2) there exists $x_0 \in C$ such that $\alpha(x_0, fx_0) \geq 1$;

(S_3) f is σ-continuous.

Then f has a fixed point, that is, there exists $z \in C$ such that $fz = z$.

Proof: By the given condition (S_2), there exists point $x_0 \in C$ such that $\alpha(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = fx_{n-1}$ for all $n \geq 0$. We split the proof into several steps.

Step-I: $\alpha(x_n, x_m) \geq 1$, for all $m > n \geq 0$. We have $\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1$. Since f is triangular α-orbital admissible, then
$\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}) \geq 1$.
Thus, by induction
$$\alpha(x_n, x_m) \geq 1 \text{ for all } m > n \geq 0.$$

Step-II: Now we need to prove that
$$\lim_{n\to\infty}\sigma(x_n, x_{n+1}) = 0.$$
By Step-I, we have $\alpha(x_n, x_m) \geq 1$ for all $m > n \geq 0$. Then, from (2.1)
$$\zeta\big(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), M(x_{n-1}, x_n)\big)$$
$$= \zeta\big(\alpha(x_n, x_{n-1})\sigma(fx_{n-1}, fx_n), M(x_{n-1}, x_n)\big) \geq 0 \quad (2.3)$$
where
$$M\big(\sigma(x_{n-1}, x_n)\big) =$$
$$= max\left\{\begin{array}{c} \sigma(x_{n-1}, x_n), \sigma(x_{n-1}, fx_{n-1}), \sigma(x_n, fx_n), \\ \frac{\sigma(x_{n-1}, fx_n) + \sigma(fx_{n-1}, x_n)}{2}, \\ \frac{[1+\sigma(x_n, fx_n)]\sigma(x_{n-1}, fx_{n-1})}{1+\sigma(fx_{n-1}, fx_n)} \end{array}\right\}$$
$$= max\left\{\begin{array}{c} \sigma(x_{n-1}, x_n), \sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \\ \frac{\sigma(x_{n-1}, x_n) + 3\sigma(x_n, x_{n+1})}{2}, \\ \frac{[1+\sigma(x_n, x_{n+1})]\sigma(x_{n-1}, x_n)}{1+\sigma(x_n, x_{n+1})} \end{array}\right\}$$
$$\leq max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\} \qquad (2.4)$$
If $max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\} = (x_n, x_{n+1})$, for all $n \in \mathbb{N}^*$ and condition (2.3) becomes
$$0 \leq \zeta\big(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \sigma(x_n, x_{n+1})\big)$$
$$< \sigma(x_n, x_{n+1}) - \alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1})$$
$$< 0,$$
which is a contradiction. Thus,
$max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\} = \sigma(x_{n-1}, x_n)$, for all n. So
$$0 \leq \zeta\big(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)\big)$$
$$< \sigma(x_{n-1}, x_n) - \alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}) \quad (2.5)$$
Consequently, we derive that for all $n \geq 1$.
$$\sigma(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1})$$
$$< \sigma(x_{n-1}, x_n) \qquad (2.6)$$
Which implies that $\{\sigma(x_n, x_{n+1})\}$ is a non-decreasing and bounded from below by zero, so there exists $K \geq 0$ such that
$$\lim_{n\to\infty}\sigma(x_n, x_{n+1}) = K. \qquad (2.7)$$
Suppose that $K > 0$, letting $s_n = \alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1})$ and $t_n = \sigma(x_{n-1}, x_n)$, using ($\zeta_3$),
$$0 \leq \limsup_{n\to\infty}\zeta\big(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)\big) < 0$$
which is a contradiction. Then we conclude that K= 0, that is,
$$\lim_{n\to\infty}\sigma(x_n, x_{n+1}) = 0. \qquad (2.8)$$

Step-III: Now, we shall prove that x_n is a σ-Cauchy sequence. Suppose to the contrary. Then, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k

$$\sigma(x_{m(k)}, x_{n(k)}) \geq \epsilon \quad (2.9)$$

Moreover, corresponding to $n(k)$ we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (2.9). Then

$$\sigma(x_{m(k)-1}, x_{n(k)}) < \epsilon \quad (2.10)$$

Using (2.9) and (2.10) and (σ_3), we get

$$\epsilon \leq \sigma(x_{m(k)}, x_{n(k)})$$
$$\leq \sigma(x_{m(k)}, x_{m(k)-1}) + \sigma(x_{m(k)-1}, x_{n(k)})$$
$$< \sigma(x_{m(k)}, x_{m(k)-1}) + \epsilon$$

By (2.8), and letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \sigma(x_{m(k)}, x_{n(k)}) = \epsilon \quad (2.11)$$

By using (2.8), (2.11), and the triangular inequality, we deduce

$$\lim_{n\to\infty} \sigma(x_{m(k)-1}, x_{n(k)-1}) = \epsilon \quad (2.12)$$
$$\lim_{n\to\infty} \sigma(x_{m(k)}, x_{n(k)-1}) = \epsilon \quad (2.13)$$
$$\lim_{n\to\infty} \sigma(x_{m(k)-1}, x_{n(k)}) \quad (2.14)$$

If $x_n = x_m$ for some $n < m$, then $x_{n+1} = fx_n = fx_m = x_{m+1}$ it follows from (2.6).

$$0 < \sigma(x_n, x_{n+1}) = \sigma(x_m, x_{m+1}) < \sigma(x_{m-1}, x_m) < \cdots\cdots < \sigma(x_n, x_{n+1}),$$

which is a contradiction. Then $x_n \neq x_m$ for all $n < m$. By (2.1) and as $\alpha(x_{m(k)-1}, x_{n(k)-1}) \geq 1$ for all $k \geq 1$, we get

$$0 \leq \zeta\left(\alpha(x_{m(k)-1}, x_{n(k)-1})\sigma(x_{m(k)}, x_{n(k)}), M(x_{m(k)-1}, x_{n(k)-1})\right)$$

where

$$M(x_{m(k)-1}, x_{n(k)-1}) = max\left\{\begin{array}{c}\sigma(x_{m(k)-1}, x_{n(k)-1}), \sigma(x_{m(k)-1}, x_{m(k)}), \sigma(x_{n(k)-1}, x_{n(k)}),\\ \frac{\sigma(x_{m(k)-1}, x_{n(k)}) + \sigma(x_{m(k)}, x_{n(k)-1})}{2}\\ \frac{[1+\sigma(x_{n(k)-1}, x_{n(k)})]\sigma(x_{m(k)-1}, x_{m(k)})}{1+\sigma(x_{m(k)}, x_{n(k)})}\end{array}\right\}$$

Letting $k \to \infty$, by using (2.7) and (2.10), we get

$$M(x_{m(k)-1}, x_{n(k)-1}) = \{\epsilon, 0, 0, \epsilon, 0\} = \epsilon.$$

The condition (ζ_3) implies that

$$0 \leq \limsup_{k\to\infty} \zeta(\alpha(x_{m(k)-1}, x_{n(k)-1})\epsilon, \epsilon)$$
$$< \epsilon - \alpha(x_{m(k)-1}, x_{n(k)-1})\epsilon < 0,$$

which is a contradiction. It follows that $\{x_n\}$ is a σ- Cauchy sequence in C. Since C is a σ-closed subset of the σ-complete (X, σ), there exists some $z \in C$, such that $\lim_{n\to\infty}\sigma(x_n, z) = 0$.

By σ-continuity of f, we get $\lim_{n\to\infty}\sigma(fx_n, fz) = 0$. On the other hand, we have

$$\sigma(z, fz) \leq \sigma(z, fx_n) + \sigma(fx_n, fz)$$

Passing to the limit as $n \to \infty$ in the above inequality, we obtain $\sigma(z, fz) = 0$, which implies z is a fixed point of f.

Theorem 2.2. Let C be a σ-closed subset of a metric-like space (X, σ) where Let $f: C \to C$ be an α-orbital admissible Z-contraction with respect to ζ. Suppose that the following conditions hold:

(S_1) f is triangular α- orbital admissible;

(S_2) there exists $x_0 \in C$ such that $\alpha(x_0, fx_0) \geq 1$;

(S_3) If $\{x_n\}$ is a sequence in C such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to \infty$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then f has a fixed point.

Proof. Following the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ such that $\alpha(x_n, x_m) \geq 1$ for all $m > n \geq 0$. Also $\{x_n\}$ is σ-Cauchy in C and converges to some $z \in C$. We claim that z is a fixed point of f. If there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} = z$ or $x_{n(k)+1} = fz$ for all k, then $\sigma(fz, z) = \sigma(x_{n(k)+1}, z)$ for all k, Letting $k \to \infty$, we get $\sigma(fz, z) = 0$, and the proof is complete. So without loss of generality, we may suppose that $x_n \neq z$ and $x_n \neq fz$ for all nonnegative integer n. Suppose that $\sigma(fz, z) > 0$. By assumption (S_3) , there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},, z) \geq 1$ for all k. By (2.1) and as $\alpha(x_{n(k)}, z) \geq 1$ for all $k \geq 1$, we get

$$\zeta(\alpha(z, x_{n(k)})\sigma(fz, x_{n(k)+1}), M(x_{n(k)}, z)) \geq 0$$

From the condition (ζ_1)

$$0 < M(x_{n(k)}, z) - \alpha(z, x_{n(k)})\sigma(fz, x_{n(k)+1}) \quad (2.15)$$

where

$$M(x_{n(k)}, z) = max\left\{\begin{array}{c}\sigma(x_{n(k)}, z), \sigma(z, fz), \sigma(x_{n(k)}, x_{n(k)+1}),\\ \frac{\sigma(x_{n(k)}, fz) + \sigma(z, x_{n(k)+1})}{2},\\ \frac{[1+\sigma(z, fz)]\sigma(x_{n(k)}, x_{n(k)+1})}{1+\sigma(x_{n(k)+1}, z)}\end{array}\right\}$$

by using (2.8), we have

$$\sigma(fz, z) \leq \liminf \sigma(fz, x_{n(k)+1}) \leq \limsup M(x_{n(k)}, z)$$
$$\leq \limsup_{n\to\infty} max\left\{\begin{array}{c}\sigma(x_{n(k)}, z), \sigma(z, fz), \sigma(x_{n(k)}, x_{n(k)+1}),\\ \frac{\sigma(x_{n(k)}, fz) + \sigma(z, x_{n(k)+1})}{2},\\ \frac{[1+\sigma(z, fz)]\sigma(x_{n(k)}, x_{n(k)+1})}{1+\sigma(x_{n(k)+1}, z)}\end{array}\right\}$$
$$= \sigma(z, fz).$$

Then

$$\lim_{n\to\infty} \sigma(x_{n(k)+1}, z) = \sigma(z, fz). \quad (2.17)$$

Also

$$\lim_{n\to\infty} M(x_{n(k)}, z) = \sigma(z, fz). \quad (2.18)$$

From (2.15), (2.17) and the condition (ζ_3), we get

$$0 \leq \limsup_{k\to\infty} \zeta\left(\alpha(z, x_{n(k)})\sigma(fz, x_{n(k)+1}), M(x_{n(k)}, z)\right) < 0$$

which is contradiction. Then we conclude that $\sigma(fz, z) = 0$ and so z is fixed point of f.

Now, we prove a uniqueness fixed point result. For this, we need the following additional condition.

($\Im$): For all $x, y \in Fix(f)$, we have $\alpha(x, y) \geq 1$, where $Fix(f)$ denotes the set of fixed points of f.

Theorem 2.3. Adding condition($\Im$) to the hypotheses of Theorem 2.1 (resp. Theorem 2.2), we obtain that z is the unique fixed point of f.

Proof: Suppose by contradiction. That is, there exist $z, w \in X$ such that $z = fz$ and $w = fw$ with $z \neq w$. By assumption $(\mathfrak{I})$, we have $\alpha(z,w) \geq 1$. So, by (2.1) and using the condition(ζ_2), we get

$$0 \leq \zeta\big(\alpha(z,w)\sigma(fz,fw), M(z,w)\big) = \zeta\big(\alpha(z,w)\sigma(z,w), M(z,w)\big) \qquad (2.19)$$

where

$$M(z,w) = max\left\{\begin{array}{c}\sigma(z,w), \sigma(z,fz), \sigma(w,fw),\\ \dfrac{\sigma(z,fw)+\sigma(w,fz)}{2},\\ \dfrac{[1+\sigma(w,fw)]\sigma(z,fz)}{1+\sigma(fz,fw)}\end{array}\right\} = \sigma(z,w),$$

Therefore, by (2.19) and (ζ_3) we get

$$0 \leq \zeta\big(\alpha(z,w)\sigma(fz,fw), M(z,w)\big) = \zeta\big(\alpha(z,w)\sigma(z,w), \sigma(z,w)\big) < \sigma(z,w) - \alpha(z,w)\sigma(z,w) < 0$$

which is a contradiction. Hence, $z = w$.

Corollary 2.1. Let C be a σ-closed subset of a metric-like space (X,σ) and let $f: C \to C$ be a given mapping. All the hypotheses of Theorem 2.1 are satisfied and suppose that $M(x,y) = \sigma(x,y)$ in (2.1). Then f has a fixed point.

Proof: The proof follows easily when taking $M(x,y) = \sigma(x,y)$ in Theorem 2.1.

Corollary 2.2. Let C be a σ-closed subset of a metric-like space (X,σ) and let $f: C \to C$ be a given mapping. All the hypotheses of Theorem 2.2 are satisfied and suppose that $M(x,y) = \sigma(x,y)$ in (2.1). Then f has a fixed point.Then f has a fixed point.

Proof: The proof follows easily when taking $M(x,y) = \sigma(x,y)$ in Theorem 2.2.

Corollary 2.3. Let C be a σ-closed subset of a metric-like space (X,σ) and let $f: C \to C$ be a given mapping. All the hypotheses of Theorem 2.1 are satisfied and suppose that $M(x,y) = \sigma(x,y)$ in (2.1). Also suppose that there exists $k \in (0,1)$ and $\alpha: C \times C \to [0,\infty)$, such that

$$\alpha(x,y)\sigma(fx,fy) \leq kM(x,y) \qquad (2.22)$$

for all $x, y \in C$, satisfying $\alpha(x,y) \geq 1$ and $M(x,y)$ is defined as earlier in (2.2). Then f has a fixed point.

Proof: The proof follows easily when taking simulation function $\zeta(t,s) = ks - t$ for all $t, s \geq 0$ in Theorem 2.1.

Corollary 2.4. Let C be a σ-closed subset of a metric-like space (X,σ) and let $f: C \to C$ be a given mapping. All the hypotheses of Theorem 2.1 are satisfied and suppose that $M(x,y) = \sigma(x,y)$ in (2.1). Also suppose that there exists $k \in (0,1)$, such that

$$\sigma(fx,fy) \leq kM(x,y) \qquad (2.23)$$

for all $x, y \in C$, and $M(x,y)$ is defined as earlier in (2.2). Then f has a fixed point.

Proof: The proof follows easily when $\alpha(x,y) = 1$, and taking simulation function $\zeta(t,s) = ks - t$ for all $t, s \geq 0$ in Theorem 2.2.

Illustrations

We present the suitable examples to illustrate the usability of the obtained results.

Example 3.1. Consider $X = \{0,1,2\}$. Take the metric-like space $\sigma: X \times X \to \mathbb{R}_0^+$ defined by

$$\sigma(0,0) = 0, \quad \sigma(0,1) = \sigma(1,0) = \tfrac{3}{8}, \sigma(0,2) = \sigma(2,0) = \tfrac{12}{23},$$
$$\sigma(1,2) = \sigma(2,1) = \frac{3}{4}, \ \ \sigma(1,1) = \frac{2}{3}, \ \ \sigma(2,2) = \frac{4}{7},$$

Note that $\sigma(2,2) \neq 0$, so σ *is not a* metric and $\sigma(2,2) > \sigma(0,2)$, so σ is not a partial metric. Clearly (X,σ) is a complete metric- like space, given $f: X \to X$ as $f0 = f1 = 0$ and $f2 = 1$. Mapping as $\alpha: X \times X \to [0,\infty]$ by

$$\alpha(x,y) = \begin{cases} 1, & if\ x, y \epsilon\ [0,2] \\ 0, & otherwise. \end{cases}$$

Next, let $\zeta: X \times X \to \mathbb{R}$, be given by $\zeta(t,s) = \frac{6}{7}s - t$.

First, let $x, y\ \epsilon\ X$ such that $\alpha(x,y) \geq 1$, by the definition of α, this implies that $x = 0$ and since $f0 = 0$, so $\alpha(fx,fy) = 1$ for each $y \epsilon X$, that is f is α-orbital admissible.

Clearly,

$$\zeta\big(\alpha(0,0)\sigma(f0,f0), M(x,y)\big) = \zeta\big(\sigma(0,0), M(0,0)\big) = 0.$$

Now the problem is categorized into five Steps.

Step I: When $x = 0$ and $y = 1$, then

$$M(0,1) = max\left\{\frac{3}{8}, 0, \frac{3}{8}, \frac{3}{16}, 0\right\} = \frac{3}{8}$$

By using the contraction condition (2.1), we have

$$\zeta\big(\alpha(0,1)\sigma(fx,fy), M(x,y)\big) = \zeta\big(\sigma(0,0), M(0,1)\big) = \zeta\left(0, \frac{3}{8}\right) = \left(\frac{6}{7}\right)\left(\frac{3}{8}\right) > 0$$

Step II: When $x = 1$ and $y = 2$, then

$$M(1,2) = max\left\{\frac{3}{4}, \frac{3}{8}, \frac{3}{4}, \frac{41}{69}, \frac{7}{16}\right\} = \frac{3}{4}$$

By (2.1), we get

$$\zeta\big(\alpha(1,2)\sigma(f1,f2)\ M(x,y)\big) = \zeta\big(\sigma(2,0), M(1,2)\big) = \zeta\left(\tfrac{12}{23}, \tfrac{3}{4}\right) = \left(\tfrac{6}{7}\right)\left(\tfrac{3}{4}\right) - \tfrac{12}{23} = \tfrac{34}{322} > 0$$

Step III: When $x = 1$ and $y = 1$, then

$$M(1,1) = max\left\{\frac{2}{3}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{33}{64}\right\} = \frac{3}{8}$$

By (2.1), we get

$$\zeta\big(\alpha(1,1)\sigma(f1,f1), M(1,1)\big) = \zeta\big(\sigma(2,2), M(1,1)\big) = \zeta\left(\tfrac{4}{7}, \tfrac{3}{8}\right) = \left(\tfrac{6}{7}\right)\left(\tfrac{3}{8}\right) - \tfrac{4}{7} > 0$$

Step IV: When $x = 2$ and $y = 2$, then

$$M(2,2) = max\left\{\frac{4}{7}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{63}{80}\right\} = \frac{63}{80}$$

By (2.1), we get

$$\zeta\big(\alpha(2,2)\sigma(f2,f2), M(x,y)\big) = \zeta\big(\sigma(0,0), M(2,2)\big) = \zeta\left(0, \tfrac{63}{80}\right) = \tfrac{63}{80} > 0$$

Step V: When $x = 0$ and $y = 2$, then

$$M(0,2) = max\left\{\frac{12}{23}, 0, \frac{3}{4}, \frac{165}{368}, 0\right\} = \frac{3}{4}$$

By (2.1), we have

$$\zeta(\alpha(0,2)\sigma(f0, f2), M(x,y)) = \zeta(\sigma(0,0), M(0,2)) = \frac{3}{4}.$$

Therefore from the above all cases contraction condition (2.1) is verified. Note that f is σ-continuous. Moreover, there exists $x_0 \epsilon X$ such that $\alpha(x_0, fx_0) \geq 1$. In fact, for $x_0 = 1$. *we* have $\alpha(1, f1) = \alpha(1,0) = 1$. Hence all hypotheses of Theorem 2.1 are satisfied. Here '0' is the fixed point of f.

Example 3.2. Let $X = [01,]$ and $\sigma: X \times X \to \mathbb{R}_0^+$ is a metric-like space defined by $\sigma(x,y) = \max\{x,y\}$, Define the mapping $f: X \times X$ by $fx = \frac{x}{1+x}$. By [10], we get f is a Z-contraction with respect to $\zeta \in Z$ where $\zeta(t,s) = \frac{s}{1+s} - t$ for all $t, s \in [0, \infty)$.

Therefore, for all $t, s \in X$, we get

$$\begin{aligned} 0 \leq \zeta(\sigma(fx, fy), \sigma(x,y)) &= \frac{\sigma(x,y)}{1+\sigma(x,y)} - \sigma(fx, fy) \\ &\leq \frac{M(x,y)}{1+M(x,y)} - \sigma(fx, fy) \\ &= \zeta(\sigma(fx, fy), M(x,y)). \end{aligned}$$

This shows that f turns into a Z-contraction with respect to ζ.

Application to Dynamic Programming

Bellman and Lee [3] first studied the existence of solutions for functional equations. Present we study the existence of solutions of functional equations and system of functional equations arising in dynamic programming have been studied by using various fixed point theorems. The reader can refer to [3, 5, 6] for a more detailed explanation of the above background. In this section, we will prove the existence of a common solution for classes of functional equations using Corollary 2.4.

Solving the following two functional equations arising in dynamic programming:

$$r(x) = \sup_{y \in D}\{g(x,y) + G(x, y, r(\tau(x,y)))\} \tag{4.1}$$

$$r(x) = \sup_{y \in D}\{g(x,y) + Q(x, y, r(\tau(x,y)))\} \tag{4.2}$$

For all $x \in W$, Where $\tau : W \times D \to W, g : W \times D \to \mathbb{R}$ and $G.Q : W \times D \times \mathbb{R} \to \mathbb{R}$. while $W \subseteq U$ is a state space and $D \subseteq V$ is a decision space and U and $V$are banach spaces. Here, we study the existence of $h^* \in B(w)$ is a common solution of the functional equations (4.1) and (4.2).

Let $B(W)$ denote the set of all the bounded real-valued functions on W. It is well known that $B(W)$ endowed with the metric-like space.

$$\sigma_b(h,k) = \sup_{y\in D} |h(x) - k(x)|, h, k \in B(W). \tag{4.3}$$

is a complete metric-like space.

Now, we consider the operators $A_1, A_2 : B(W) \to B(W)$, and $x \in W$ defined by

$$A_1(h)(x) = \sup_{y\in D}\{g(x,y) + G(x, y, h(\tau(x,y)))\}, \tag{4.4}$$

and

$$A_2(h)(x) = \sup_{y\in D}\{g(x,y) + Q(x, y, h(\tau(x,y)))\}, \tag{4.5}$$

these mappings are well-defined if the functions g, G and Q are bounded. Also denote

$$M(h_1, h_2) == max\left\{\begin{array}{c} \sigma(h_1,h_2), \sigma(h_1, fh_1), \sigma(h_2, fh_2), \\ \frac{\sigma(h_1, fh_2) + \sigma(fh_1, h_2)}{2}, \\ \frac{[1+\sigma(h_2, fh_2)]\sigma(h_1, fh_1)}{1+\sigma(fh_1, fh_2)} \end{array}\right\}$$

for all $h_1, h_2 \in B(W)$.

Theorem 4.1 Suppose that there exists $k \in (0,1)$ such that for every $(x,y) \in W \times D$ *and* $h_1, h_2 \in B(W)$, the inequality

$$\left|G(x, y, h_1(\tau(x,y))) - Q(x, y, h_2(\tau(x,y)))\right| \leq k\, M(h_1, h_2).$$

holds. Then A_1, A_2 have a common fixed point in (BW).

Proof: Let $\lambda > 0$ be an arbitrary positive real number, $x \epsilon W$ *and* $h_1, h_2 \epsilon B(W)$. Then by (4.4) and (4.5), there exist $y_1, y_2 \epsilon D$ *such that*

$$A_1(h_1)(x) < g(x, y_1) + G(x, y_1, h_1(\tau(x, y_1))) + \lambda \tag{4.6}$$

$$A_2(h_2)(x) < g(x, y_2) + Q(x, y_2, h_2(\tau(x, y_2))) + \lambda \tag{4.7}$$

$$A_1(h_1)(x) \geq g(x, y_2) + G(x, y_2, h_1(\tau(x, y_2))) \tag{4.8}$$

and

$$A_2(h_2)(x) \geq g(x, y_1) + Q(x, y_1, h_2(\tau(x, y_1))) \tag{4.9}$$

Then from (4.7) and (4.8), it follows easily that

$$\begin{aligned} &A_1(h_1)(x) - A_2(h_2)(x) \\ &\leq G(x, y_1, h_1(\tau(x, y_1))) + \lambda - Q(x, y_1, h_2(\tau(x, y_1))) \\ &\leq \left|G(x, y_1, h_1(\tau(x, y_1))) - Q(x, y_1, h_2(\tau(x, y_1)))\right| \\ &\leq kM(h_1, h_2) + \lambda. \end{aligned}$$

Similarly, from (4.6) and (4.9), we get

$$A_2(h_2)(x) - A_1(h_1)(x) \leq kM(h_1, h_2) + \lambda.$$

We deduce from above inequalities that

$$|A_1(h_1)(x) - A_2(h_2)(x)| \leq kM(h_1, h_2) + \lambda \tag{4.10}$$

Since the inequality (4.10) is true for any $x \epsilon W$, then

$$\sigma(A(h_1), B(h_2) \leq kM(h_1, h_2) + \lambda. \tag{4.11}$$

Again $\lambda > 0$ is arbitrary, so

$$\sigma(A(h_1), B(h_2)) \leq kM(h_1, h_2) \tag{4.12}$$

So Corollary 2.4 is applicable. Consequently, the mappings A and B have a common fixed point.

Note that, the functional equations (4.1) and (4.2) has a common solution $h^* \epsilon B(W)$.

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