International Journal of Advanced Technology & Engineering Research (IJATER)
International Conference on "Recent Advancement in Science & Technology" (ICRAST 2017)

APPLICATION OF G-FUNCTION TO SOLVE THE PROBLEM RELATED TO COOLING OF A SPHERE

DR. S. S. SHRIVASTAVA[1], DR. PINKI SIKARWAR[2]
Department of Mathematics,
Institute for Excellence in Higher Education[1], Radha Raman Institute of Technology & Science[2],
Bhopal (M.P.)

ABSTRACT

We have observed that the separation of variables method can be applied to several types of problems on bounded spatial domains. The problem must be linear and have homogeneous boundary conditions. In this paper we present the solution to a classical problem in three dimensions, the cooling of a sphere. The assumed symmetries in the problem will permit us to reduce the dimension of the problem to one spatial dimension and time.

INTRODUCTION

The Meijer's [3] G-function of one variable was defined in terms of Mellin-Barnes type integrals as follows:

$$G_{p,q}^{m,n}\left[x\Big|_{(b_j,1)_{1,q}}^{(a_j,1)_{1,p}}\right] = \frac{1}{2\pi i}\int_L \theta(s)\, x^s ds \qquad (1)$$

where i = √(– 1),

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}$$

and an empty product is interpreted as 1, $0 \le m \le q$, $0 \le n \le p$, and the parameters are such that no poles of $\Gamma(b_j - s)$ (j = 1, …, m) coincides with any pole of $\Gamma(1 - a_j + s)$ (j = 1, …, n).

There are three different paths L of integration:

(i) L runs from – i ∞ to + i ∞ so that all poles of $\Gamma(b_j - s)$ (j = 1, …, m) are to the right and all the poles of $\Gamma(1 - a_j + s)$ (j = 1, …, n) to the left of L. The integral converges if p + q < 2 (m + n) and |arg x| < (m + n – ½ p – ½ q)π.

(ii) L is a loop starting and ending at + ∞ and encircling all poles of $\Gamma(b_j - s)$ (j = 1, …, m) once in the negative direction, but none of the poles of $\Gamma(1 - a_j + s)$ (j = 1, …, n). The integral converges if q ≥ 1 and either p < q or p = q and |x| < 1.

(iii) L is a loop starting and ending at – ∞ and encircling all poles of $\Gamma(1 - a_j + s)$ (j = 1, …, n) once in the positive direction, but none of the poles of $\Gamma(b_j - s)$ (j = 1, …, m). The integral converges if p ≥ 1 and either p > q or p = q and |x| < 1.

Given a sphere whose initial temperature depends only on the distance from the center (e.g., a constant initial temperature) and whose boundary is kept at a constant temperature, predict the temperature at any point inside the sphere at a later time. This is the problem, for example, of determining the temperature of the center of a potato that has been put in a hot oven. The reader might also conjecture that this problem is important for medical examiners who want to determine the time of an individual's death. Early researchers, notably Kelvin, used this problem to determine the age of the earth based on assumptions about its initial temperature and its temperature today.

This cooling problem may remind the reader of Newton's law of cooling, which is encountered in ordinary differential equations texts. Recall that this law states that the rate at which a body cools is proportional to the difference of its temperature and the temperature of the environment. Quantitatively, if *T* = T(t) is the temperature of a body and T_e is the temperature of its environment, then $T'(t) = K(T_e - T)$, where K is the constant of proportionality. But the reader should note that this law applies only in the case that the body has a uniform, homogeneous temperature. In the PDE problem we are considering, the temperature may vary radically throughout the body.

In this paper, we shall make application of following modified form of the integral [1]:

$$\int_0^{\pi} (\sin x)^{s-1} \sin nx \, dx = \frac{\pi \sin \frac{n\pi}{2} \Gamma(s)}{2^{s-1} \Gamma\left(\frac{s \pm n+1}{2}\right)} \qquad (2)$$

where, Re (s) > 0and Legendre’s duplication formula

$$\sqrt{\pi}\, \Gamma(2z) = 2^{2z-1} \Gamma(z)\, \Gamma(z + \tfrac{1}{2}). \qquad (3)$$

INTEGRAL

The integral, which we need as follows:

$$\int_0^{\pi} (\sin x)^{\omega-1} \sin nx \; G^{m,l}_{p,q}\left[z\,(\sin x)^2 \Big|^{(a_j,1)_{1,p}}_{(b_j,1)_{1,q}} \right] dx$$

$$= \sqrt{\pi}\, \sin \frac{n\pi}{2}\, G^{m,l+2}_{p+2,q+2}\left[z \Big|^{\left(1-\frac{\omega}{2},1\right),\left(\frac{1}{2}-\frac{\omega}{2},1\right),(a_j,1)_{1,p}}_{(b_j,1)_{1,q},\left(\frac{1}{2}-\frac{\omega}{2}\pm\frac{n}{2},1\right)} \right], \qquad (4)$$

provided that $\omega > 0$, Re[w + 2$\min_{1\le j\le m} b_j$] > 0, |arg z| < ($m + l - \tfrac{1}{2} p - \tfrac{1}{2} q$)π.

Proof:

The integral (4) can be obtained easily by making use of the definition of G-function as given in (1) and the formulae (2) and (3).

FORMULATE THE PROBLEM

For simplicity, let us consider a sphere of radius ρ = π whose initial temperature is T_0 = constant. We will assume that the boundary is held at zero degrees for all time t >0. If *u* is the temperature, then in general *u* will depend on three spatial coordinates and time. But a little reflection shows that the temperature will depend only on the distance from the center of the sphere and on time. Evidently, the temperature *u* must satisfy the heat equation

$$u_t = k\Delta u,$$

where *k* is the diffusivity and Δ is the Laplacian. It should be clear that a spherical coordinate system ρ, ϕ, θ is more appropriate than a rectangular system, and $u = u(\rho, t)$. Because *u* does not depend on the angles ϕ and θ, the Laplacian takes on a particularly simple form:

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho}$$

Therefore, we may formulate the model as

$$u_t = k\left(u_{\rho\rho} + \frac{2}{\rho} u_\rho\right),\ 0 \le \rho < \pi,\ t > 0. \qquad (5)$$

$$u(\pi, t) = 0,\ t > 0, \qquad (6)$$

$$u(\rho, 0) = T_0,\ 0 \le \rho < \pi. \qquad (7)$$

Observe that there is an implied boundary condition at $\rho = 0$, namely that the temperature should remain bounded.

To solve problem (5)–(7) we assume $u(\rho, t) = y(\rho)g(t)$. Substituting into the PDE and separating variables gives, following solution, which is given in [2, p. 144 – 146]:

$$u(\rho, t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2 kt} \frac{\sin n\rho}{\rho} \qquad (8)$$

where

$$c_n = \frac{2}{\pi} \int_0^{\pi} T_0\, \rho \sin n\rho \, d\rho \qquad (9)$$

SOLUTION OF THE PROBLEM

The solution of the problem to be obtained is

$$u(\rho, t) = \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} e^{-n^2 kt} \frac{\sin n\rho}{\rho}\, \sin \frac{n\pi}{2}$$

$$\times\, G^{m,l+2}_{p+2,q+2}\left[z \Big|^{\left(1-\frac{\omega}{2},1\right),\left(\frac{1}{2}-\frac{\omega}{2},1\right),(a_j,1)_{1,p}}_{(b_j,1)_{1,q},\left(\frac{1}{2}-\frac{\omega}{2}\pm\frac{n}{2},1\right)} \right], \qquad (10)$$

where all conditions of convergence are same as in (2).

Proof:

Choose

$$f(\rho) = u(\rho, 0) = T_0 = \rho^{-1} (\sin \rho)^{\omega-1} G^{m,l}_{p,q}\left[z\,(\sin\rho)^2 \Big|^{(a_j,1)_{1,p}}_{(b_j,1)_{1,q}} \right]. \qquad (11)$$

Now combining (11) and (9) and making the use of the integral (4), we derive

$$c_n = \frac{2}{\sqrt{\pi}}\, sin\frac{n\pi}{2} G^{m,l+2}_{p+2,q+2}\left[z \Big|^{\left(1-\frac{\omega}{2},1\right),\left(\frac{1}{2}-\frac{\omega}{2},1\right),(a_j,1)_{1,p}}_{(b_j,1)_{1,q'}\left(\frac{1}{2}-\frac{\omega}{2}\pm\frac{n}{2},1\right)} \right], \quad (12)$$

Putting the value of C_n from (12) in (8), we get the required result (10).

SPECIAL CASES

On specializing the parameters, G-function may be reduced to several other higher transcendental functions. Therefore the result (10) is of general nature and may reduced to be in different forms, which will be useful in the literature on applied Mathematics and other branches.

REFERENCES

[1] Gradshteyn, I. S. & Ryzhik's, I. M.: Table of Integrals, Series and Products, Academic Press, New York (1980), 372.
[2] J. David Logan: Partial Differential Equations: Springer, 2004.
[3] Meijer, C. S.: Ibidem 49 (1946), p. 344-456.