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SOME FIXED POINT AND COMMON FIXED POINT THEOREMS IN L- SPACES FOR INTEGRAL TYPE MAPPINGS

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Abstract

In the present paper some fixed point and common fixed point results are established for integral type mappings for L-spaces. The result satisfies various known results.

Key Words: Fixed Point, Common Fixed point Integral type mappings

Introduction

It was shown by S. Kashara [7] in 1975 that several known generalization of the Banach contraction theorem can be derived easily from a fixed point theorem in an L-space. Iseki [5] has used the fundamental idea of Kashara to investigate the generalization of some known fixed point theorem in L-space. Many other mathematicians Yeh [16], Singh [13], Pachpatte [9], Pathak and Dubey [10], Patel, Sahu and Sao [11], Patel and Patel [12], Som [15], Sao [16] worked for L- spaces. In the present chapter a similar investigation for the study of fixed point and common fixed point theorems in L-spaces are worked out. We find some fixed point and common fixed point theorems in L-spaces for integral type mappings motivated by [1].

3. This paper is divided into two sections.

Section I: Some fixed point theorems in L-spaces for Integral type Mappings

Section II: Some common fixed point theorems in L-spaces for Integral type Mappings

In both sections we will find some new results in L-spaces **4.**

Preliminaries

Definition (4.A): L -Space: Let N be a set of all non negative integers and X is a non-empty set.A pair $(X,\to)$ of a set X and a subset $\to$ of the set $X^N\times X$, is called an L -space if

(i) If $x_n = x$, where $x\in X$, for all $n\in N$, then $\left(\{x_n\}_{n\in N}, x\right)\in\to$

(ii) $\left(\{x_n\}_{n\in N}, x\right)\in\to$, then $\left\{\{x_{n_i}\}_{i\in N}, x\right\}\in\to$, for every $\{x_{n_i}\}_{i\in N}$ of $\{x_n\}_{n\in N}$ in what follows, we shall write $\{x_n\}_{n\in N}\to x$, or $x_n\to x$ instead of $\left(\{x_n\}_{n\in N}, x\right)\in\to$ and read $\{x_n\}_{n\in N}$ converges to x.

Definition (4.B): An L -Space $(X,\to)$ is said to be separated if each sequence in X converge to at most one point of X .

Definition (4.C): A mapping T of an L -Space $(X,\to)$ into an L -Space $(X,\to)$ is said to be continuous if $x_n\to x \Rightarrow Tx_n\to Tx$, For some sub sequence $\left\{x_{n_i}\right\}_{i\varepsilon N}, \left\{x_n\right\}_{n\varepsilon N}$

Definition (4.D):Let d be non-negative extended real valued function on $X\times X$, $0\le d(x,y)<\infty$, for all $x,y\in X$, an L -Space $(X,\to)$ is said to be L -complete if each sequence, $\{x_n\}_n\in N$, in X with $\sum d(x_i, y_{i+1})<\infty$ converge to at most one point of X .

Lemma (4.E): **(K.S.):** Let $(X,\to)$ be an L -Space which is d -complete for a non-negative real valued function d on $X\times X$, if $(X,\to)$ is separated, then $d(x,y)=d(y,x)=0$, implies $x=y$ for every x,y in X.

Theorem (4.F) (Branciari)[2] Let (X, d) be a complete metric space, c ∈ 0,1 and let f: X → X be a mapping such that for each x, y ∈ X,

$$\int_0^{d(fx,fy)}\xi(t)dt \le c\int_0^{d(fx,fy)}\xi(t)dt \quad \text{---- (4.F)}$$

Where ξ: [0, +∞) → [0, +∞) is a Lesbesgue integrable mapping which is sum able on each compact subset of [0, +∞)

, non negative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \xi(t)dt$, then f has a unique fixed point a ∈ X such that for each x ∈ X, $\lim_{n\to\infty} f^n x = a$. After the paper of Branciari, a lot of a research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [13] extending the result of Brianciari by replacing the condition **(4.F)** by the following

$$\int_0^{d(fx,fy)} \xi(t)dt \le \int_0^{max\,d\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}} \xi(t)dt$$

In [3] the author proved the following

Theorem B([3]) Let (X, d) be a complete metric space and f : X → X such that

$$\int_0^{d(fx,fy)} u(t)dt \le \alpha \int_0^{\{d(x,fx)+d(y,fy),\}} u(t)dt + \beta \int_0^{d(x,y)} u(t)dt + \gamma \int_0^{max\{d(x,fy)+d(y,fx),\}} u(t)$$

For each x, y Є X with non negative reals α, β, γ such that $2\alpha + \beta + 2\gamma < 1$, where

u: [0, + ∞) →[0, +∞) is a Lebesgue integrable mapping which is sum able, non-negative and such that for each Є > 0, $\int_0^{\epsilon} u(t)dt > 0$.
Then f has a unique fixed point in X.

There is a gap in the proof of the theorem B. In fact, the authors [3] used the inequality

$\int_0^{a+b} u(t)dt \le \int_0^{a} u(t)dt + \int_0^{b} u(t)dt \; for\; 0 \le a < b,$
This is not true in general.This was modified bu [4]

Also we are using effect $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ which is only true when f (2a – x) = f(x), there is again a gap, to complete our proof that $\int_0^{kf(x)} \phi(t)dt = k\int_0^{f(x)} \phi(t)dt$ which is not true in general.

We will find some fixed point theorems in L-spaces for integral type mappings in the section I.

SECTION I: SOME FIXED POINT THEOREMS IN L-SPACES FOR INTEGRAL TYPE MAPPINGS

THEOREM (4.1.1): Let $(X,\to)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x)=0$ for all x in X. Let E be a continuous shelf map of X, satisfying the conditions:

(4.1.1 a)

$$\int_0^{[d(Ex,Ey)]^2} \xi(t)\,dt \le \alpha\int_0^{d(x,Ex)d(y,Ey)} \xi(t)dt + \gamma\int_0^{d(y,Ey)d(y,Ex)} \xi(t)dt$$

$$+\delta\int_0^{d(x,Ey)d(y,Ex)} \xi(t)dt + \eta\int_0^{\max\{d(x,Ex)d(y,Ey),d(x,Ex)d(y,Ex),d(y,Ey)d(y,Ex),d(x,Ey)d(y,Ex)\}} \xi(t)dt$$

$\forall x,y\in X, \alpha,\gamma,\delta,\eta\ge 0, with\, \alpha+\delta+\eta \prec 1. Then\, E\, has\, a\, unique fixed\, point.$
Where $\xi:[0,\infty]\to[0,\infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0,\infty]$, non negative, and such that, $\forall\varepsilon \succ 0\int_0^{\varepsilon}\xi(t)dt \succ 0$.

PROOF: Let x_0 be an arbitrary point in X; define sequence $\{x_n\}$ recurrently,

$$Ex_0 = x_1, \ldots\ldots\ldots\ldots\ldots Ex_n = x_{n+1}, \qquad \text{Where, } n = 0,1,2,3\ldots\ldots\ldots$$

Now by (4.1.1 a),we have

$$\int_0^{[d(x_1,x_2)]^2} \xi(t)dt = \int_0^{[d(Ex_0,Ex_1)]^2} \xi(t)dt$$
$$\le \alpha\int_0^{d(x_0,Ex_0)d(x_1,Ex_1)} \xi(t)dt$$
$$+\gamma\int_0^{d(x_1,Ex_1)d(x_1,Ex_0)+\delta d(x_0,Ex_1)d(x_1,Ex_0)} \xi(t)dt$$
$$+\eta\int_0^{\max\{d(x_0,Ex_0),d(x_1,Ex_1),d(x_0,Ex_0),d(x_1,Ex_0),d(x_1,Ex_1)d(x_1,Ex_0)}$$

$$\leq \alpha\int_0^{d(x_0,x_1)d(x_1,x_2)+\beta d(x_0,x_1)d(x_1,x_1)} \xi(t)dt + \gamma\int_0^{d(x_1,x_2)d(x_1,x_1)+\delta d(x_0,x_2)d(x_1,x_1)} \xi(t)dt$$
$$+\eta\int_0^{\max\{d(x_0,x_1),d(x_1,x_2),d(x_0,x_1),d(x_1,x_1),d(x_1,x_2),d(x_1,x_1),d(x_0,x_2),d(x_1,x_1)\}} \xi(t)dt$$

$$\int_0^{[d(x_1,x_2)]^2} \xi(t)dt \leq [\alpha+\eta]\int_0^{d(x_0,x_1)d(x_1,x_2)} \xi(t)dt$$

$$\int_0^{[d(x_1,x_2)]} \xi(t)dt \leq [\alpha+\eta]\int_0^{d(x_0,x_1)} \xi(t)dt$$

Similarly

$$\int_0^{[d(x_2,x_3))]} \xi(t)dt \leq k\int_0^{d(x_1,x_2)} \xi(t)dt \leq k.k\int_0^{d(x_0,x_1)} \xi(t)dt$$

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$$\int_0^{d(x_n,x_{n+1})} \xi(t) \leq k^n\int_0^{d(x_0,x_1)} \xi(t)dt,\ where\, k=[\alpha+\eta]$$

for every natural number we can say that $\sum d(x_n,x_{n+1}) \leq \infty$

By d -completeness of the space, the sequence $\{E^n x_0\}, n\in N$ converges to some u in X .By continuity of E ,the sub sequence $\{E^{n_i} x_0\}$, also converges to u.

$$\lim_{i\to\infty} E^{n_{i+1}} x_0 = E_u$$
$$\lim_{i\to\infty} E^{n_i} x_0 = u$$
$$E(\lim_{i\to\infty} E^{n_i} x_0) = E\,u$$
$$\lim_{i\to\infty} E^{n_{i+1}} x_0 \;\; = Eu$$

$\Rightarrow \;\; E_u = u$, so u is a fixed point of E.

Uniqueness: In order to prove that u is the unique fixed point of E, if possible let V be any other fixed point of $E(v\neq u)$. Then

$$d(u,v) = d(Eu,Ev)$$

$$\int_0^{[d(Eu,Ev)]^2} \xi(t)dt \leq \alpha\int_0^{d(u,Eu)d(v,Ev)} \xi(t)dt + \beta\int_0^{d(u,Eu)d(v,Ev)} \xi(t)dt + \gamma\int_0^{d(v,Ev)d(v,Eu)} \xi(t)dt + \delta\int_0^{d(u,Ev)d(v,Eu)} \xi(t)dt$$
$$+\eta\int_0^{\max\{d(u,Eu),d(v,Ev),d(u,Eu),d(v,Eu),d(v,Ev),d(v,Eu),d(u,Ev),d(v,Eu)\}} \xi(t)dt$$

$$\int_0^{[d(u,v)]^2} \xi(t)dt \leq [\delta+\eta]\int_0^{[d(u,v)]^2} \xi(t)dt$$

This is a contradiction because $\delta+\eta<1$. So E has a unique fixed point in X .

Now we will prove another fixed point theorem which is stronger then theorem (4.1.1)

THEOREM (4.1.2): Let $(X,\rightarrow)$ be a separated L -Space which is d -complete for a non-negative real valued function d on $X\times X$ with $d(x,x)=0$ for all x in X . Let E be a continuous shelf map of X ,satisfying the conditions:

(4.1.2 a)

$$\int_0^{[d(Ex,Ey)]} \xi(t)dt \leq \alpha\int_0^{[d(x,Ex)d(y,Ey)+d(x,Ex)d(y,Ex)+d(y,Ey)d(y,Ex)+d(x,Ey)d(y,Ex)]^{\frac{1}{2}}} \xi(t)dt$$

$\forall x,y\in X$ *and* $0\prec\alpha\prec 1$, *then* E *has a unique fixed point*.
Where $\xi:[0,\infty]\rightarrow[0,\infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0,\infty]$, non negative, and such that, $\forall\varepsilon\succ 0\int_0^{\varepsilon}\xi(t)dt\succ 0$.

PROOF: Let x_0 be an arbitrary point in X ,define sequence $\{x_n\}$ recurrently,

$Ex_0 = x_1, ...Ex_n = x_{n+1},$ Where, $n=0,1,2,3.........$

Now by (4.1.2 a) we have

$$\int_0^{d(x_1,x_2)} \xi(t)dt = \int_0^{d(Ex_0,Ex_1)} \xi(t)dt$$

$$\leq \alpha\int_0^{[d(x_0,Ex_0)d(x_1,Ex_1)+d(x_0,Ex_0)d(x_1,Ex_0)+d(x_1,Ex_1)d(x_1,Ex_0)+d(x_0,Ex_1)d(x_1,Ex_0)]^{\frac{1}{2}}} \xi(t)dt$$

$$= \alpha\int_0^{[d(x_0,x_1)d(x_1,x_2)+d(x_0,x_1)d(x_1,x_1)+d(x_1,x_2)d(x_1,x_1)+d(x_0,x_2)d(x_1,x_1)]^{\frac{1}{2}}} \xi(t)dt$$

$$\int_0^{d(x_1,x_2)} \xi(t)dt \leq \alpha\int_0^{[d(x_0,x_1)d(x_1,x_2)]^{\frac{1}{2}}} \xi(t)dt$$

$$\int_0^{d(x_1,x_2)} \xi(t)dt \leq \alpha^2\int_0^{[d(x_0,x_1)]} \xi(t)dt$$

Similarly,

$$\int_0^{d(x_2,x_3)} \xi(t)dt \le \alpha^2 \int_0^{[d(x_1,x_2)]} \xi(t)dt = \alpha^2.\alpha^2 \int_0^{[d(x_0,x_1)]} \xi(t)dt$$

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$$\int_0^{d(x_n,x_{n+1})} \xi(t)dt \le k^n \int_0^{[d(x_0,x_1)]} \xi(t)dt,$$

$$where\, k = \alpha^2$$

For every natural number we can say that $\sum d(x_n, x_{n+1}) \le \infty$

By d -completeness of the space, the sequence $\{E^n x_0\}$, $n \in N$ converges to some u in X. By continuity of E, the sub sequence $\{E^{n_i} x_0\}$ also converges to u.

$$\lim_{i\to\infty} E^{n_{i+1}} x_0 = E_u$$

$$\lim_{i\to\infty} E^{n_i} x_0 = u$$

$$E(\lim_{i\to\infty} E^{n_i} x_0) = E\,u$$

$$\lim_{i\to\infty} E^{n_{i+1}} x_0 \quad = Eu$$

$\Rightarrow$ E_u =u, so u is a fixed point of E.

Uniqueness: In order to prove that u is the unique fixed point of E, if possible let V be any other fixed point of E, $(v \ne u)$. Then

$$d(u,v) = d(Eu, Ev)$$

$$\int_0^{d(Eu,Ev)} \xi(t)dt \le \alpha \int_0^{\left[d(u,Eu)d(v,Ev)+d(u,Eu)d(v,Ev)+d(v,Ev)d(v,Eu)+d(u,Ev)d(v,Eu)\right]^{\frac{1}{2}}} \xi(t)dt$$

$$\int_0^{d(u,v)} \xi(t)dt \le \alpha^2 \int_0^{d(u,v)} \xi(t)dt$$

This is a contradiction because $\alpha < 1$.So E has a unique fixed point in X.

Now we are finding some common fixed point theorems for two, three and four mappings in section II.

SECTION II: SOME COMMON FIXED POINT THEOREMS IN L-SPACES FOR INTEGRAL TYPE MAPPINGS

THEOREM (4.2.1): Let $(X, \to)$ be a separated L -Space which is d -complete for a non-negative real valued function d on $(X \times X)$ with $d(x,x) = 0$ for all x in X. Let E and T be two continuous shelf mappings of X, satisfying the conditions:

$$(4.2.1a) \quad ET = TE, E(X) \subseteq T(X)$$

(4.2.1 b) $$\int_0^{[d(Ex,Ey)]^2} \xi(t)dt \le \alpha \int_0^{d(Tx,Ex)d(Ty,Ey)+\beta d(Tx,Ex)d(Ty,Ex)} \xi(t)dt + \gamma \int_0^{d(Ty,Ey)d(Ty,Ex)+\delta d(Tx,Ey)d(Ty,Ex)} \xi(t)dt$$

$$+\eta \int_0^{\max\{d(Tx,Ex)d(Ty,Ey), d(Tx,Ex)d(Ty,Ex), d(Ty,Ey)d(Ty,Ex), d(Tx,Ey)d(Ty,Ex)\}} \xi(t)dt$$

$$\forall\, x, y \in X, \alpha, \beta, \gamma, \delta, \eta \ge 0, with\, \alpha + \delta + \eta \prec 1.$$

Then E and T have a unique common fixed point. Where $\xi : [0,\infty] \to [0,\infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0,\infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^{\varepsilon} \xi(t)dt \succ 0$.

PROOF: Let x_0 be an arbitrary point in X,since

$E(x) \subseteq T(x),$ we can choose $x_1 \in X$ such that $Ex_0 = Tx_1, Ex_1 = Tx_2, ...Ex_n = Tx_{n+1}$

For $n = 1, 2, 3..............$

$$\int_0^{[d(Tx_{n+1},Tx_{n+2})]^2} \xi(t)dt = \int_0^{[d(Ex_n,Ex_{n+1})]^2} \xi(t)dt$$

$$\le \alpha \int_0^{d(Tx_n,Ex_n)d(Tx_{n+1},Ex_{n+1})+\beta d(Tx_n,Ex_n)d(Tx_{n+1},Ex_n)} \xi(t)dt$$

$$+\gamma \int_0^{d(Tx_{n+1},Ex_{n+1})d(Tx_{n+1},Ex_n)+\delta d(Tx_n,Ex_{n+1})d(Tx_{n+1},Ex_n)} \xi(t)dt$$

$$+\eta \int_0^{\max\{d(Tx_n,Ex_n)d(Tx_{n+1},Ex_{n+1}), d(Tx_n,Ex_n)d(Tx_{n+1},Ex_n), d(Tx_{n+1},Ex_{n+1})d(Tx_{n+1},Ex_n), d(Tx_n,Ex_{n+1})d(Tx_{n+1},Ex_n)\}} \xi(t)dt$$

$$= \alpha \int_0^{d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+2})+\beta d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+1})} \xi(t)dt$$
$$+\gamma \int_0^{d(Tx_{n+1},Tx_{n+2})d(Tx_{n+1},Tx_{n+1})+\delta d(Tx_n,Tx_{n+2})d(Tx_{n+1},Tx_{n+1})} \xi(t)dt$$
$$+\eta \int_0^{\max\{d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+2}),d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+1}),d(Tx_{n+1},Tx_{n+2})d(Tx_{n+1},Tx_{n+1}),d(Tx_n,Ex_{n+1})d(Tx_{n+1},Tx_{n+1})\}} \xi(t)dt$$

$$\int_0^{d(Tx_{n+1},Tx_{n+2})} \xi(t)dt \le [\alpha+\eta] \int_0^{d(Tx_n,Tx_{n+1})} \xi(t)dt$$

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Hence,

$$\int_0^{d(Tx_{n+1},Tx_{n+2})} \xi(t)dt \le k^n \int_0^{d(Tx_1,Tx_2)} \xi(t)dt, \text{where}\, k = \alpha+\eta$$

for every natural number m ,we can write the

$$\sum_m^\infty d(x_m, x_{m+1}) < \infty$$

By d -completeness of x ,the sequence $\{T^n x_0\}_{n\in N}$ converges to some $u \in X$. Since $E(x) \subseteq T(x),$ therefore the subsequence t of $\{T^n x_0\}$ such that, $E(T(u)) \to Eu,$ and $T(E(u)) \to Tu$

So we have, $Eu = Tu$

Since $\lim_{n\to\infty} T^n X_0 = u, T(\lim_{n\to\infty} T^n X_0) = Tu$
………… (4.2.1c)

This implies that $Tu = u$

Hence $Tu = Eu = u$

Thus u is common fixed point of E and T.

Uniqueness: For the uniqueness of the common fixed point, if possible let v be any other common fixed point of E and T ; Then from (4.2.1b)

$$d(u,v) = d(Eu, Ev)$$

$$\int_0^{[d(Eu,Ev)]^2} \xi(t)dt \le \alpha \int_0^{d(Tu,Eu)d(Tv,Ev)} \xi(t)dt + \beta \int_0^{d(Tu,Eu)d(Tv,Ev)} \xi(t)dt$$
$$+\gamma \int_0^{d(Tv,Ev)d(Tv,Eu)} \xi(t)dt + \delta \int_0^{d(Tu,Ev)d(Tv,Eu)} \xi(t)dt$$
$$+\eta \int_0^{\max\{d(Tu,Eu)d(Tv,Ev),d(Tu,Eu)d(Tv,Eu),d(Tv,Ev)d(Tv,Eu),d(Tu,Ev)d(Tv,Eu)\}} \xi(t)dt$$

$$\int_0^{[d(u,v)]^2} \xi(t)dt \le [\delta+\eta] \int_0^{[d(u,v)]^2} \xi(t)dt$$

Which is a contradiction because $[\delta+\eta] < 1$

Hence E and T have unique common fixed point in X

THEOREM (4.2.2): Let $(X, \to)$ be a separated L -Space which is d -complete for a non-negative real valued function d on $X \times X$ with $d(x, x=)0$ for all x in X .Let E and T be two continuous shelf mappings of X ,satisfying the conditions:

$$(4.2.2a) \quad ET = TE, E(X) \subseteq T(X)$$

$$(4.2.2b) \int_0^{d(Ex,Ey)} \xi(t)dt \le \alpha \int_0^{[d(Tx,Ex)d(Ty,Ey)+d(Tx,Ex)d(Ty,Ex)+d(Ty,Ey)d(Ty,Ex)+d(Tx,Ey)d(Ty,Ex)]^{\frac{1}{2}}} \xi(t)dt$$

$\forall x, y\, \varepsilon\, X, \alpha \ge 0,$
$with\, \alpha \prec 1. Then\, E\, and\, T\, have\, a\, unique\, common\, fixed\, point.$

Where $\xi : [0,\infty] \to [0,\infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0,\infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^\varepsilon \xi(t)dt \succ 0$.

PROOF: Let x_0 be an arbitrary point in x, since $E(x) \subseteq T(x)$

We can chose $x_1 \in X$ such that $Ex_0 = Tx_1, Ex_1 = Tx_2 ....Ex_n = Tx_{n+1}$

for $n = 1, 2, 3............$

$$\int_0^{d(Tx_{n+1},Tx_{n+2})} \xi(t)dt = \int_0^{d(Ex_n,Ex_{n+1})} \xi(t)dt$$

$$\le \alpha \int_0^{[d(Tx_n,Ex_n)d(Tx_{n+1},Ex_{n+1})+d(Tx_n,Ex_n)d(Tx_{n+1},Ex_n)+d(Tx_{n+1},Ex_{n+1})d(Tx_{n+1},Ex_n)+d(Tx_n,Ex_{n+1})d(Tx_{n+1},Ex_n)]^{\frac{1}{2}}} \xi(t)dt$$

$$= \alpha\int_0^{[d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+2})+d(Tx_n,Tx_{n+1})d(Tx_{n+1},Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})d(Tx_{n+1},Tx_{n+1})+d(Tx_n,Tx_{n+2})d(Tx_{n+1},Tx_{n+1})]^{\frac{1}{2}}} \xi(t)dt$$

$$\int_0^{d(Tx_{n+1},Tx_{n+2})} \xi(t)dt \leq [\alpha]\int_0^{[d(Tx_n,Tx_{n+1})]^{\frac{1}{2}}} \xi(t)dt$$

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Hence, $d(Tx_{n+1},Tx_{n+2}) \leq k^n\, d(Tx_1,Tx_2), where\, k = \alpha^2$

For every natural number m, we can write the

$$\sum_m^{\infty} d(x_m, x_{m+1}) < \infty$$

By d-completeness of X, the sequence $\{T^n x_0\}_{n\in N}$ converges to some $u \in X$. Since $E(x) \subseteq T(x),$ So $E(T(u)) \to Eu$, and $T(E(u)) \to Tu$

we have, $Eu = Tu$

Since $\lim_{n\to\infty} T^n X_0 = u, T(\lim(n\to\infty) T^n X_0) = Tu$ …………… (4.2.2b)

This implies that $Tu = u$ Hence $Tu = Eu = u$

Thus u is common fixed point of E and T.

Uniqueness: For the uniqueness of the common fixed point, if possible let v be any other common fixed point of E and T; Then from (4.2.2b)

$$d(u,v) = d(Eu, Ev)$$

$$\int_0^{d(Eu,Ev)} \xi(t)dt \leq \delta\int_0^{[d(Tu,Eu)d(Tv,Ev)+d(Tu,Eu)d(Tv,Ev)+d(Tv,Ev)d(Tv,Eu)+d(Tu,Ev)d(Tv,Eu)]^{\frac{1}{2}}} \xi(t)dt$$

$$\int_0^{d(u,v)} \xi(t)dt \leq \alpha\int_0^{[d(u,v)]^{\frac{1}{2}}} \xi(t)dt$$

Which is a contradiction because $\alpha < 1$.

Hence E and T have unique common fixed point in X.

In next theorems we will prove the common fixed point theorems for three mappings.

THEOREM -(4.2.3) Let $(X, \to)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x) = 0$ for all x in X. Let E, F, T be three continuous shelves mapping of X, satisfying the conditions:

(4.2.3 a) $ET = TE, FT = TF,\ E(X) \subset T(X)\ And\ F(X) \subset T(X)$

(4.2.3 b)
$$\int_0^{[d(Ex,Fy)]^2} \xi(t)dt \leq \alpha\int_0^{d(Tx,Ex)d(Ty,Fy)} \xi(t)dt + \gamma\int_0^{d(Ty,Fy)d(Ty,Ex)} \xi(t)dt + \delta\int_0^{d(Tx,Fy)\,d(Ty,Ex)} \xi(t)dt + \eta\int_0^{\max[\alpha\, d(Tx,Ex)d(Ty,Fy),\beta\, d(Tx,Ex)d(Ty,Ex),d(Ty,Fy)d(Ty,Ex),d(Tx,Fy)\,d(Ty,Ex)]} \xi(t)dt$$

For all $x, y \in X$, and $\alpha, \gamma, \eta, \delta \geq 0$ with $\alpha + \delta + \eta < 1.$ Then E, F, T have unique common fixed point. Where $\xi : [0,\infty] \to [0,\infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0,\infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^{\varepsilon} \xi(t)dt \succ 0$.

PROOF: Let x_0 be a point in X .Since $E(X) \subset T(X)$, we can choose a point x_1 in X such that $Tx_1 = Ex_0,$ also $F(X) \subset T(X)$. We can choose a point x_2 in x such that $Tx_2 = Fx_1$.

In general we can choose the point

$Tx_{2n+1} = Ex_{2n},$ ………..... (4.2.3d)

$Tx_{2n+2} = Fx_{2n+1},$. …………… (4.2.3e)

For every $n \in N,$ we have

$$\int_0^{[d(Tx_{2n+1},Tx_{2n+2})]^2} \xi(t)dt = \int_0^{[d(Ex_{2n},Fx_{2n+1})]^2} \xi(t)dt$$

$$\int_0^{[d(Ex,Fy)]^2} \xi(t)dt \leq \alpha\int_0^{d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Fx_{2n+1})} \xi(t)dt + \beta\int_0^{d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Ex_{2n})} \xi(t)dt + \gamma\int_0^{d(Tx_{2n+1},Fx_{2n+1})d(Tx_{2n+1},Ex_{2n})} \xi(t)dt + \delta\int_0^{d(Tx_{2n},Fx_{2n+1})\,d(Tx_{2n+1},Ex_{2n})} \xi(t)dt + \eta\int_0^{\max[d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Fx_{2n+1}),d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Ex_{2n}),d(Tx_{2n+1},Fx_{2n+1})d(Tx_{2n+1},Ex_{2n}),d(Tx_{2n},Fx_{2n+1})\,d(Tx_{2n+1},Ex_{2n})]} \xi(t)dt$$

$$\int_0^{d(Tx_{2n+1},Tx_{2n+2})}\xi(t)dt \le [\alpha+\eta]\int_0^{d(Tx_{2n},Tx_{2n+1})}\xi(t)dt$$

For $n=1,2,3,............,$

$$\int_0^{d(Tx_{2n+1},Tx_{2n+2})}\xi(t)dt \le k\int_0^{d(Tx_{2n},Tx_{2n+1})}\xi(t)dt\text{, where } k=\alpha+\eta$$

Similarly we have

$$\int_0^{d(Tx_{2n+1},Tx_{2n+2})}\xi(t)dt \le k^n\int_0^{d(Tx_1,Tx_0)}\xi(t)dt$$

$$\sum_{i=o}^{\infty} d(Tx_{2i+1},Tx_{2i+2}) < \infty$$

Thus the d-completeness of the space implies the sequence $\{T^n x_0\}_{n\in N}$, converges to some u in X.

So by (4.2.3d) and (4.2.3 e), $(E^n x_0)_{n\in N}$, and $(F^n x_0)_{n\in N}$ also converges to the some point u respectively.

Since E, T and F are continuous, there is a subsequence t of $\{T^n x_0\}_{n\in N}$, such that $E(T(t))\to Eu, T(E(t))\to Tu, F(T(t))\to Fu$ and $T(F(t))\to Tu$

Hence we have $Eu = Fu = Tu$ ……………..(3.2.3 f)

Thus
$T(Tu)=T(Eu)=E(Fu)=T(Fu)=F(Tu)=E(Tu)=F(Eu)=F(Fu)$ …………(4.2.3 g)

Now if $Eu \ne F(Eu)$

$$\int_0^{[d(Eu,F(Eu))]^2}\xi(t)dt \le \alpha\int_0^{d(Tu,Eu)d(TEu,FEu)}\xi(t)dt + \gamma\int_0^{d(TEu,FEu)d(TEu,Eu)}\xi(t)dt + \delta\int_0^{d(Tu,FEu)\,d(TEu,Eu)}\xi(t)dt + \eta\int_0^{\max[\alpha\, d(Tu,Eu)d(TEu,FEu),\, d(Tu,Eu)d(TEu,Eu),\, d(TEu,FEu)d(TEu,Eu),\, d(Tu,FEu)\,d(TEu,Eu)]}\xi(t)dt$$

$$\int_0^{[d(Eu,F(Eu))]^2}\xi(t)dt \le [\delta+\eta]\int_0^{d(Tu,F(Eu))\,d(Tu,F(Eu))}\xi(t)dt$$

$$\int_0^{[d(Eu,F(Eu))]}\xi(t)dt \le [\delta+\eta]\int_0^{d(Eu,F(Eu))}\xi(t)dt$$

Which is a contradiction because $\delta+\eta<1$

Hence $Eu = F(Eu)$ …………….. (4.2.3 h)

From (4.2.3 g) & (4.2.3 h) we have

$Eu = F(Eu) = Tu = E(Eu)$

Hence Eu is a common fixed point of E, F & T

Uniqueness: Let u & v $(u\ne v)$ be two common fixed points of $E, F \,\&\, T$

Then by (4.2.3 b), we have

$d(u,v) = d(Eu, Fv)$

$$\int_0^{[d(Eu,Fv)]^2}\xi(t)dt \le \alpha\int_0^{d(Tu,Eu)d(Tv,Fv)}\xi(t)dt + \gamma\int_0^{d(Tv,Fv)d(Tv,Eu)}\xi(t)dt + \delta\int_0^{d(Tu,Fv)\,d(Tv,Eu)}\xi(t)dt + \eta\int_0^{\max[\alpha\, d(Tu,Eu)d(Tv,Fv),\, \beta\, d(Tu,Eu)d(Tv,Eu),\, d(Tv,Fv)d(Tv,Eu),\, d(Tu,Fv)\,d(Tv,Eu)]}\xi(t)dt$$

$$\int_0^{[d(u,v)]^2}\xi(t)dt \le [\delta+\eta]\int_0^{[d(u,v)]^2}\xi(t)dt$$

Which is a contradiction, because $\delta+\eta<1$

Hence $u = v$

So $E, F \,\&\, T$ have unique common fixed point.

THEOREM (4.2.4) Let $(X,\to)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X\times X$ with $d(x,x)=0$ for all x in X. Let E, F, T be three continuous shelves mapping of X, satisfying the conditions:

(3.2.4 a) $ET = TE, FT = TF,\ E(X)\subset T(X)\ And\ F(X)\subset T(X)$

(4.2.4b) $\int_0^{d(Ex,Fy)}\xi(t)dt \le \alpha\int_0^{[d(Tx,Ex)d(Ty,Fy)+d(Tx,Ex)d(Ty,Ex)+d(Ty,Fy)d(Ty,Ex)+d(Tx,Fy)\,d(Ty,Ex)]^{\frac{1}{2}}}\xi(t)dt$

For all $x, y \in X$ and $\alpha \geq 0$ with $\alpha < 1$.Then E, F, T have unique common fixed point. Where $\xi : [0, \infty] \rightarrow [0, \infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0, \infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^{\varepsilon} \xi(t)dt \succ 0$.

PROOF: Let x_0 be a point in X .Since $E(X) \subset T(X)$, we can choose a point x_1 in X such that $Tx_1 = Ex_0$, also $F(X) \subset T(X)$. We can choose a point x_2 in x such that $Tx_2 = Fx_1$.

In general we can choose the point

$$Tx_{2n+1} = Ex_{2n},$$

………………… (4.2.4c)

$$Tx_{2n+2} = Fx_{2n+1}.$$

…………………..(4.2.4d)

For every $n \in N,$ we have

$$[d(Tx_{2n+1}, Tx_{2n+2})] = [d(Ex_{2n}, Fx_{2n+1})]$$

$$\int_0^{d(Ex_{2n},Fx_{2n+1})} \xi(t)dt \leq \alpha \int_0^{[d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Fx_{2n+1})+d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Ex_{2n})+d(Tx_{2n+1},Fx_{2n+1})d(Tx_{2n+1},Ex_{2n})+d(Tx_{2n},Fx_{2n+1})\,d(Tx_{2n+1},Ex_{2n})]^{\frac{1}{2}}} \xi(t)dt$$

$$\int_0^{d(Tx_{2n+1},Tx_{2n+2})} \xi(t)dt \leq \alpha^2 \int_0^{d(Tx_{2n},Tx_{2n+1})} \xi(t)dt$$

For $n = 1, 2, 3..............,$

$$\int_0^{(Tx_{2n+1},Tx_{2n+2})} \xi(t)dt \leq k \int_0^{d(Tx_{2n},Tx_{2n+1})} \xi(t)dt\,, where\ k = \alpha^2$$

Similarly we have

$$\int_0^{(Tx_{2n+1},Tx_{2n+2})} \xi(t)dt \leq k^n \int_0^{d(Tx_1,Tx_0)} \xi(t)dt$$

$$\sum_{i=o}^{\infty} d(Tx_{2i+1}, Tx_{2i+2}) < \infty$$

Thus the d -completeness of the space implies the sequence $\{T^n x_0\}_{n \in N}$, converges to some u in X .

So $(E^n x_0)_{n \in N}$, and $(F^n x_0)_{n \in N}$, also converges to the some point u respectively.

Since E, T and F are continuous, there is a subsequence t of $\{T^n x_0\}_{n \in N}$, such that $E(T(t)) \rightarrow Eu.T(E(t)) \rightarrow Tu, F(T(t)) \rightarrow Fu$ and $T(F(t)) \rightarrow Tu.$

Hence we have $Eu = Fu = Tu$ ……………….(4.2.4 e)

Thus $T(Tu) = T(Eu) = E(Fu) = T(Fu) = F(Tu) = E(Tu) = F(Eu) = F(Fu)$
……………..(4.2.4 f)

So we have, if $Eu \neq F(Eu)$

$$\int_0^{d(Eu,F(Eu))} \xi(t)dt \leq \alpha \int_0^{[d(Tu,Eu)d(TEu,FEu)+d(Tu,Eu)d(TEu,Eu)+d(TEu,FEu)d(TEu,Eu)+d(Tu,FEu)\,d(TEu,Eu)]^{\frac{1}{2}}} \xi(t)dt$$

$$\int_0^{d(Eu,F(Eu))} \xi(t)dt \leq \alpha \int_0^{[d(Tu,F(Eu))\,d(Tu,F(Eu))]^{\frac{1}{2}}} \xi(t)dt$$

$$\int_0^{[d(Eu,F(Eu))]} \xi(t)dt \leq [\delta]^2 \int_0^{d(Eu,F(Eu))} \xi(t)dt$$

Which is a contradiction because $\delta < 1$.

Hence $Eu = F(Eu)$ ………………. (4.2.4 g)

So

$$Eu = F(Eu) = T(Eu) = E(Eu)$$

Hence Eu is a common fixed point of $E, F \& T$

Uniqueness: Let $u \& v (u \neq v)$ be two common fixed points of $E, F \& T$ Then we have

$$d(u, v) = d(Eu, Fv)$$

$$\int_0^{d(Eu,Fv)} \xi(t)dt \leq \alpha \int_0^{[d(Tu,Eu)d(Tv,Fv)+d(Tu,Eu)d(Tv,Eu)+d(Tv,Fv)d(Tv,Eu)+d(Tu,Fv)\,d(Tv,Eu)]^{\frac{1}{2}}} \xi(t)dt$$

$$d(u, v) \leq [\delta]^2\, d(u, v)$$

Which is a contradiction, because $\alpha < 1$

Hence $u = v.$

So $E, F \& T$ have unique common fixed point.

THEOREM (4.2.5) Let $(X, \rightarrow)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x) = 0$ for all x in X.

Let E, F, T be three continuous selves mapping of X satisfying (4.2.5 a) and

(4.2.5.a) $$\int_0^{[d(E^p x, F^q y)]^2} \xi(t)dt \le \alpha \int_0^{d(Tx, E^p x) d(Ty, F^q y)} \xi(t)dt + \gamma \int_0^{d(Ty, F^q y) d(Ty, E^p x)} \xi(t)dt + \delta \int_0^{d(Tx, F^q y)\, d(Ty, E^p x)} \xi(t)dt + \eta \int_0^{\max[\alpha\, d(Tx, E^p x) d(Ty, F^q y), \beta\, d(Tx, E^p x) d(Ty, E^p x), d(Ty, F^q y) d(Ty, E^p x), d(Tx, F^q y)\, d(Ty, E^p x)]} \xi(t)dt$$

For all x, y in X, $Tx \ne Ty, \alpha, \gamma, \eta, \delta \ge 0$ with $\alpha + \eta + \delta < 1$,

If some positive integer p, q exists such that E^p, F^q and T are continuous, Then E, F, T have a unique fixed point in X. Where $\xi : [0, \infty] \rightarrow [0, \infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0, \infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^{\varepsilon} \xi(t)dt \succ 0$.

PROOF - It follows from (4.2.4) $E^p T = TE^p, F^q T = TF^q, E^p(x) \subset T(x)$ ………….. (4.2.5b)

And $F^q(x) \subset T(x)$ F^q(x)⊂T(x) ………….. (4.2.5.c)

i.e. u is the fixed point of T, E^p and F^q

now $T(Eu) = E(Tu) = E(u) = E(E^p u) = E^p(Eu)$ .... (4.2.4.d)

and $T(Fu) = F(Tu) = F(u) = F(E^q u) = F^q(Fu)$ .... (4.2.4.e)

Hence it follows that Eu is common fixed point of T, E^p and Fu is a common fixed point of T and F^q. The uniqueness of u, can be proved easily.

THEOREM (4.2.6) Let $(X, \rightarrow)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x) = 0$ for all x in X.

Let E, F, T be three continuous selves mapping of X satisfying (4.2.5 a) and

$$\int_0^{d(E^p x, F^q y)} \xi(t)dt \le \alpha \int_0^{[d(Tx, E^p x) d(Ty, F^q y) + d(Tx, E^p x) d(Ty, E^p x) + d(Ty, F^q y) d(Ty, E^p x) + d(Tx, F^q y)\, d(Ty, E^p x)]^{\frac{1}{2}}} \xi(t)dt$$

…………(4.2.6 a)

For all x, y in X, $Tx \ne Ty, \alpha \ge 0$ with $\alpha < 1$,

If some positive integer p, q exists such that E^p, F^q and T are continuous,Then E, F, T have a unique fixed point in X. Where $\xi : [0, \infty] \rightarrow [0, \infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0, \infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^{\varepsilon} \xi(t)dt \succ 0$.

PROOF: It can be proved easily by the help of (4.2.5).

Now we will prove some common fixed point theorem for four mappings, which contains rational expressions.

THEOREM (4.2.7) Let $(X, \rightarrow)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x) = 0$ for all x in X.

Let E, F, T be three continuous selves mapping of X satisfying the conditions:

(4.2.7 a) $ES = SE, FT = TF, E(X) \subset T(X)$ and $F(X) \subset S(X)$

(4.2.7b) $$\int_0^{d(Ex, Fy)} \xi(t)dt \le \alpha \int_0^{\max\left[d(Sx, Ex)\left\{\frac{d(Ty, Fy) + d(Ex, Ty)}{d(Sx, Ty) + d(Ex, Ty)}\right\}, d(Sx, Ty)\right]} \xi(t)dt, \text{ for all } x, y \,\varepsilon\, X$$

with $[d(Sx, Ty) + d(Ex, Ty)] \ne 0$ and $\alpha \prec 1$, then E, F, T and S have a unique fixed point. Where $\xi : [0, \infty] \rightarrow [0, \infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0, \infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^{\varepsilon} \xi(t)dt \succ 0$.

PROOF: Let $x_0 \in X$, there exists a point $x_1 \in X$, such that $Tx_1 = Ax_0$, and for this point x_1, we can choose a point

$x_2 \in X$, such that $Bx_1 = Sx_2$ and so on inductively, we can define a sequence $\{y_n\}$ in X such that $y_{2n} = Tx_{2n+1} = Ex_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Fx_{2n+1}$,

where $n = 0,1,2..........$

we have , $$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)dt = \int_0^{d(Ex_{2n},Fx_{2n+1})} \xi(t)dt$$

$$\leq \alpha \int_0^{\max\left[d(Sx_{2n},Ex_{2n})\left\{\frac{d(Tx_{2n+1},Fx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}\right\},d(Sx_{2n},Tx_{2n+1})\right]} \xi(t)dt$$

$$= \alpha \int_0^{\max\left[d(Sx_{2n},Ex_{2n})\left\{\frac{d(Tx_{2n+1},Fx_{2n+1})+d(Tx_{2n+1},Tx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Tx_{2n+1},Tx_{2n+1})}\right\},d(Sx_{2n},Tx_{2n+1})\right]} \xi(t)dt$$

$$= \alpha \int_0^{\max[d(Tx_{2n+1},Fx_{2n+1}),d(Sx_{2n},Tx_{2n+1})]} \xi(t)dt$$

Case Ist :

$$\max\left[d(Tx_{2n+1},Fx_{2n+1}),d(Sx_{2n},Tx_{2n+1}) = d(Tx_{2n+1},Fx_{2n+1})\right]$$

Then $d(Ex_{2n},Fx_{2n+1}) \leq \alpha\, d(Ex_{2n},Fx_{2n+1})$

Which is not possible, because $\alpha < 1$. So taking

$$\max\left[d(Tx_{2n+1},Fx_{2n+1}),d(Sx_{2n},Tx_{2n+1}) = d(Sx_{2n},Tx_{2n+1})\right]$$

Hence $d(y_{2n},y_{2n+1}) \leq \alpha\, d(Sx_{2n},Tx_{2n+1})$

$$d(y_{2n},y_{2n+1}) \leq \alpha\, d(y_{2n-1},y_{2n})$$

For every integer $p > 0$, we get

$$d(y_n,y_{n+p}) \leq d(y_n,y_{n+1}) d(y_{n+1},y_{n+2}) -------+ d(y_{n+p-1},y_{n+p})$$

$$d(y_n,y_{n+p}) \leq \left\{\frac{\alpha^p}{1-\alpha}\right\} d(y_n,y_{n+1})$$

Letting $n \to \infty$, we have $d(y_n,y_{n+p}) \to 0$. Therefore $\{y_n\}$ is a Cauchy sequence in X. By d -completeness of X , $\{y_n\}_{n\in N}$, is converges to some $u \in X$. So subsequence $\{Ex_{2n}\},\{Fx_{2n+1}\},\{Tx_{2n}\}$ and $\{Sx_{2n+1}\}$ of $\{y_n\}$ also converges to same point u .Since E,F,T and S are continuous, such that

$$E[S(x_n)] \to Eu, S[E(x_n)] \to Su, F[T(x_n)] \to Fu,$$,and $$T[F(x_n)] \to Tu$$

So, $Eu = Su; Fu = Tu$

Now from (4.2.7 a) and (4.2.7b)

$$\int_0^{d(E^2x_{2n},Fx_{2n+1})} \xi(t)dt = \int_0^{d[E(Ex_{2n}),Fx_{2n+1}]} \xi(t)dt$$

$$\leq \alpha \int_0^{\max\left[d(S(Ex_{2n}),E(Ex_{2n}))\left\{\frac{d(Tx_{2n+1},Fx_{2n+1})+d(E(Ex_{2n}),Tx_{2n+1})}{d(S(Ex_{2n}),Tx_{2n+1})+d(E(Ex_{2n}),Tx_{2n+1})}\right\},d(S(Ex_{2n}),Tx_{2n+1})\right]} \xi(t)dt$$

$$= \alpha \int_0^{\max\left[d(Su,Eu)\left\{\frac{d(u,u)+d(Eu,u)}{d(Su,u)+d(Eu,u)}\right\},d(Su,u)\right]} \xi(t)dt$$

d(Eu,u) ≤ d(Su,u) = α d(Eu,u)

This is a contradiction, because $\alpha < 1$.

So $Eu = Su = u$, that is u is common fixed point of E and S . Similarly we can prove

$Fu = Tu = u$. So E,F,S and T have common fixed point.

Uniqueness: In order to prove uniqueness of common fixed point, let v be another fixed point of E,F,T and S ,such that $v \neq u$,

$$\int_0^{d(u,v)} \xi(t)dt = \int_0^{d(Eu,Fv)} \xi(t)dt \leq \alpha \int_0^{\max\left[d(Su,Eu)\left\{\frac{d(Tv,Fv)+d(Eu,Tv)}{d(Su,Tv)+d(Eu,Tv)}\right\},d(Su,Tv)\right]} \xi(t)dt$$

$d(u,v) \leq \alpha d(u,v)$ this is a contradiction because $\alpha < 1$.

Hence u is the unique common fixed point of E,T,F and S .

This completes the proof.

THEOREM (4.2.8) Let $(X, \rightarrow)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x)=0$ for all x in X.

Let E,F,T be three continuous selves mapping of X satisfying the conditions:

$$\text{(4.2.8 a)} \quad ES = SE, FT = TF, E(X) \subset T(X) \text{ and } F(X) \subset S(X)$$

$$\text{(4.2.8b)} \quad \int_0^{d(Ex,Fy)} \xi(t)dt \le \alpha \int_0^{\max\left[d(Ex,Ty)\left\{\frac{d(Sx,Ex)+d(Fy,Ty)}{d(Sx,Ty)+d(Ex,Ty)}\right\}, d(Sx,Ty)\right]} \xi(t)dt, \text{ for all } x, y \varepsilon X$$

with $[d(Sx,Ty)+d(Ex,Ty)] \ne 0$ *and* $\alpha \prec 1$, *then* E,F,T *and* S *have a unique fixed point.*

Where $\xi : [0,\infty] \rightarrow [0,\infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0,\infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^{\varepsilon} \xi(t)dt \succ 0$.

PROOF: This theorem follows by theorem (4.2.7)

THEOREM (4.2.9) Let $(X, \rightarrow)$ be a separated L-Space which is d-complete for a non-negative real valued function d on $X \times X$ with $d(x,x)=0$ for all x in X.

Let E,F,T be three continuous selves mapping of X satisfying the conditions:

$$\text{(4.2.9 a)} \quad ES = SE, FT = TF, E(X) \subset T(X) \text{ and } F(X) \subset S(X)$$

$$\text{(4.2.9 b)} \quad \int_0^{d(Ex,Fy)} \xi(t)dt \le \alpha \int_0^{\max\left[d(Sx,Ex)\left\{\frac{d(Ty,Fy)+d(Ex,Ty)}{d(Sx,Ty)+d(Ex,Ty)}\right\}, d(Ex,Ty)\left\{\frac{d(Sx,Ex)+d(Fy,Ty)}{d(Sx,Ty)+d(Ex,Ty)}\right\}, [d(Sx,Ex)+d(Fy,Ty)], d(Sx,Ty)\right]} \xi(t)dt$$

for all $x, y \varepsilon X$ *with* $[d(Sx,Ty)+d(Ex,Ty)] \ne 0$ *and* $2\alpha \prec 1$

Then E,F,T *and* S *have a unique common fixed point.*

Where $\xi : [0,\infty] \rightarrow [0,\infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0,\infty]$, non negative, and such that, $\forall \varepsilon \succ 0 \int_0^{\varepsilon} \xi(t)dt \succ 0$.

PROOF: On applying the process same as in theorem (4.2.7)

$$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)dt = \int_0^{d(Ex_{2n},Ex_{2n+1})} \xi(t)dt$$

$$\le \alpha \int_0^{\max\left[d(Sx_{2n},Ex_{2n})\left\{\frac{d(Tx_{2n+1},Fx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}\right\}, d(Ex_{2n}Tx_{2n+1})\left\{\frac{d(Sx_{2n},Ex_{2n})+d(Tx_{2n+1},Fx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}\right\}, [d(Sx_{2n},Ex_{2n})+d(Tx_{2n+1},Fx_{2n+1})], d(Sx_{2n},Tx_{2n+1})\right]}$$

$$= \alpha \int_0^{\max[d(y_{2n},y_{2n+1}),[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})],d(y_{2n-1},y_{2n})]} \xi(t)dt$$

CaseIst :

If

$$\max[d(y_{2n},y_{2n+1}),[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})],d(y_{2n-1},y_{2n})] = d(y_{2n},y_{2n+1})$$

Then by case first of theorem (4.2.7), it is a contradiction.

Case IInd

If,

$$\max[d(y_{2n},y_{2n+1}),\{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})\},d(y_{2n-1},y_{2n})] = d(y_{2n-1},y_{2n})$$

Then by theorem (4.2.7), it is clear that, E,F,T and S have unique common fixed point.

Case IIIrd

$$\max[d(y_{2n},y_{2n+1}),[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})],d(y_{2n-1},y_{2n})] = [d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})]$$

Then

$$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)dt \le \alpha \int_0^{[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})]} \xi(t)dt$$

$$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)dt \le \frac{\alpha}{1-\alpha} \int_0^{d(y_{2n-1},y_{2n})} \xi(t)dt$$

$$\int_0^{d(y_{2n},y_{2n+1})} \xi(t)dt \le k \int_0^{d(y_{2n-1},y_{2n})} \xi(t)dt, \qquad \text{where } k = \frac{\alpha}{1-\alpha} \prec 1$$

As applying the same process in second part of theorem, we get a unique common fixed point for E,F,T and S. This completes the proof.

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