International Journal of Advanced Technology & Engineering Research (IJATER)
www.ijater.com

SOME FIXED POINT RESULTS IN HILBERT SPACE FOR RATIONAL MAPPING

Pankaj Patel[1], Mahesh Tiwari[2]
1. Prof. and Head Department of Mathematics, Technocrats Institute of Technology & Excellence Bhopal
2. Research Scholar, AISECT University Bhopal
Email: pankajpateltit@ gmail.com

ABSTRACT

In this paper, some results on fixed point theorem in Hilbert Space by using self mapping are established for new rational expressions. The results will extend and generalized well-known previous results.
Keywords: Common fixed point, continuous self- mapping, Hilbert Space.
AMS Subject classification (2000): *47H10, 54H25.*

2. Introduction and Preliminaries:

After the Banach's fixed point theorems many researchers worked on Hilbert spaces for generalizing this principle. Most of them worked for random operator. In recent years, the study of random fixed points have attracted much attention some of the recent literatures in random fixed points may be noted in [1,2,3,4].In particular ,random iteration schemes leading to random fixed point of random operators have been discussed in [5,6,7]. But the present chapter deals with the basic results on Hilbert space. We find unique fixed point and common fixed point theorems in Hilbert space. Our results are motivated by Kanan [9] Koparde P.V. and Waghmode D.B. [10] Shrivastava, Salu and Nair Smita [11,12] Tiwari K. and Lahiri B.K. [13] and extended many previous known results. In order to establish the above cited theorem we define the followings:

2.1 Normed Space: A linear space X over the scalar field K (K = Real or Complex) is called Normed space if there is a function $\|.\|: X \to R^+$ satisfying:

(1) $\|x\| \geq 0 \quad \forall x \in X$

(2) $\|x\| = 0$ iff $x = 0, \ \forall x \in X$

(3) $\|x + y\| \leq \|x + z\| + \|z + y\|, \ \forall x, y, z \in X$

(4) $\|\alpha x\| = |\alpha| \|x\|, \ \forall \alpha \in K$

2.2 Inner Product Space: Let X be a vector space over the field of real or complex numbers. A mapping, denoted by $<.>$ or $(.,.)$ defined on $X \times X$ into the underlying field is called an inner product of any two elements x and y of X if the following conditions are satisfied:

(1) $< x, x > \geq 0 \ \forall x \in X$

$< x, x > = 0$ iff $x = 0 \ \forall x \in X$

(2) $< x + y, z > = < x, z > + < y, z > \ \forall x, y, z \in X$

(3) $< \alpha x, y > = \alpha < x, y > \ \forall \alpha \in K$

(4) $< \overline{x, y} > = < y, x > \ \forall x, y \in K$

2.3 Hilbert Space: An inner product space X is called a Hilbert space if the norm space induced by the inner product is a Banach space (complete norm space). That is every Cauchy sequence $\{x_n\} \in X$ with respect to the norm induced by the inner product is convergent with respect to this norm.

2.4 Convex Set: A set A in a normed space is said to be convex if

$$\alpha x + (1 - \alpha) y \in A \quad \forall x, y \in A, \ \alpha \in [0,1]$$

2.5 Fixed Point: Let H be a Hilbert space and $T: H \to H$ be a mapping . A point x is said to be a fixed point of T if $T(x) = x$.

2.6 Common Fixed Point: Let H be a Hilbert space and $S, T: H \to H$ be two mappings. A point x is said to be a common fixed point of S and T if $S(x) = T(x) = x$.

2.7 Sharma *et.al* [13] have proved the following a common fixed point theorem for self –mapping satisfying the following condition

$$\|Tx - Ty\| \leq \alpha \frac{\|x - Tx\|^2 + \|y - Ty\|^2}{\|x - Tx\| + \|y - Ty\|} + \beta \|x - y\|$$

For all $x, y \in C, x \neq y, \ 0 \leq \alpha < \frac{1}{2}, 0 \leq \beta, \ 2\alpha + \beta < 1.$

MAIN RESULTS:

In this section, we prove a common unique fixed point theorem for self-mapping in Hilbert spaces.

Theorem 3. 1: Let C be a non empty closed subset of Hilbert space H

Let $T: C \to C$ be a self -mapping satisfies the following Condition

$$\|Tx - Ty\| \leq \alpha \frac{\|x - Tx\| + \|y - Ty\| + \|x - y\|}{1 + \|x - Tx\| \|x - Ty\| \|y - Ty\| \|x - y\|}$$

$$+\beta\frac{\|x-Ty\|^2+\|y-Tx\|^2}{\|x-Ty\|+\|y-Tx\|}+\gamma\|x-y\|$$

For all $x,y\in C, x\neq y,\ 0\leq\alpha,\beta<\frac{1}{2}\ 0\leq\gamma$ and $3\alpha+2\beta+\gamma<1$.

Then T has a unique fixed point in C.

Proof: Let C be a non-empty closed subset of Hilbert space H and $T: C\rightarrow C$ be a self –mapping.

Let $x_0\in C$ is an arbitrary. We define a sequence $\{x_n\}$ in C as follows

$x_{n+1}=Tx_n=T^{n+1}x_0$, for $n=0,1,2,3,\dots\dots\dots\dots\dots$

If for some $n,\ x_n=x_{n+1}$

Then it is immediately follows x_n is a fixed point of T.

Now we suppose that $x_{n+1}\neq x_n$, for $n=0,1,2,3,\dots\dots\dots\dots\dots$

For all $n\geq 1$ we have

$$\|x_{n+1}-x_n\|=\|Tx_n-Tx_{n-1}\|$$
$$\leq\alpha\frac{\|x_n-Tx_n\|+\|x_{n-1}-Tx_{n-1}\|+\|x_n-x_{n-1}\|}{1+\|x_n-Tx_n\|\|x_n-Tx_{n-1}\|\|x_{n-1}-Tx_{n-1}\|\|x_n-x_{n-1}\|}$$
$$+\beta\frac{\|x_n-Tx_{n-1}\|^2+\|x_{n-1}-Tx_n\|^2}{\|x_n-Tx_{n-1}\|+\|x_{n-1}-Tx_n\|}+\gamma\|x_n-x_{n-1}\|$$
$$=\alpha\frac{\|x_n-x_{n+1}\|+\|x_{n-1}-x_n\|+\|x_n-x_{n-1}\|}{1+\|x_n-x_{n+1}\|\|x_n-x_n\|\|x_{n-1}-x_n\|\|x_n-x_{n-1}\|}$$
$$+\beta\frac{\|x_n-x_n\|^2+\|x_{n-1}-x_{n+1}\|^2}{\|x_n-x_n\|+\|x_{n-1}-x_{n+1}\|}+\gamma\|x_n-x_{n-1}\|$$
$$\leq\alpha\{\|x_n-x_{n+1}\|+2\|x_{n-1}-x_n\|\}+\beta\|x_{n-1}-x_{n+1}\|+\gamma\|x_n-x_{n-1}\|$$
$$\Rightarrow(1-\alpha-\beta)\|x_{n+1}-x_n\|\leq(2\alpha+\beta+\gamma)\|x_n-x_{n-1}\|$$
$$\Rightarrow\|x_{n+1}-x_n\|\leq\left(\frac{2\alpha+\beta+\gamma}{1-\alpha-\beta}\right)\|x_n-x_{n-1}\|$$
$$\Rightarrow\|x_{n+1}-x_n\|\leq s\|x_n-x_{n-1}\|$$

Where $s=\left(\frac{2\alpha+\beta+\gamma}{1-\alpha-\beta}\right)<1$ since $3\alpha+2\beta+\gamma<1$.

Now since $\|x_{n+1}-x_n\|\leq s\|x_n-x_{n-1}\|\leq s^2\|x_{n-1}-x_{n-2}\|\leq\ \dots\dots\dots\leq s^n\|x_1-x_0\|$

$i.e., \|x_{n+1}-x_n\|\leq s^n\|x_1-x_0\|$ For all $n\geq 1$.

Now we will prove that $\{x_n\}$ is a Cauchy sequence for this

$$\|x_n-x_{n+p}\|\leq\|x_n-x_{n+1}\|+\|x_{n+1}-x_{n+2}\|+\dots\dots\dots+\|x_{n+p-1}-x_{n+p}\|$$
$$\leq(s^n+s^{n+1}+s^{n+2}+\cdots\dots\dots+s^{n+p-1})\|x_1-x_0\|$$
$$=s^n(1+s+s^2+\cdots\dots\dots+s^{p-1})\|x_1-x_0\|$$
$$\leq\left(\frac{s^n}{1-s}\right)\|x_1-x_0\|$$

$\Rightarrow\|x_n-x_{n+p}\|\rightarrow 0$ as $n\rightarrow\infty$, since $p<1$.

Therefore $\{x_n\}$ is a Cauchy sequence.

Since C is a closed subset of Hilbert space.

Therefore there exist an element $u\in C$ such that $\lim_{n\rightarrow\infty}x_n=u$.

Now we will prove that u is a fixed point of T.

Let $Tu\neq u$.

Consider $\|u-Tu\|=\|u-x_n+x_n-Tu\|$
$$\leq\|u-x_n\|+\|x_n-Tu\|$$
$$=\|x_n-Tu\|$$
$$=\|Tx_{n-1}-Tu\|$$
$$\leq\alpha\frac{\|x_{n-1}-Tx_{n-1}\|+\|u-Tu\|+\|x_{n-1}-u\|}{1+\|x_{n-1}-Tx_{n-1}\|\|x_{n-1}-Tu\|\|u-Tu\|\|x_{n-1}-u\|}$$
$$+\beta\frac{\|x_{n-1}-Tu\|^2+\|u-Tx_{n-1}\|^2}{\|x_{n-1}-Tu\|+\|u-Tx_{n-1}\|}+\gamma\|x_{n-1}-u\|$$
$$Taking\ n\rightarrow\infty\ \Rightarrow\|u-Tu\|\leq 0$$
$$\Rightarrow u=Tu$$

Hence u is a fixed point of T.

Uniqueness: Suppose that v be the another fixed point of T such that

.

Then $\|u-v\|=\|Tu-Tv\|$
$$\leq\alpha\frac{\|u-Tu\|+\|v-Tv\|+\|u-v\|}{1+\|u-Tu\|\|u-v\|\|v-Tv\|}$$
$$+\beta\frac{\|u-Tv\|^2+\|v-Tu\|^2}{\|u-Tv\|+\|v-Tu\|}+\gamma\|u-v\|$$
$$\leq\alpha\{\|u-Tu\|+\|v-Tv\|+\|u-v\|\}+2\beta\{\|u-v\|\}+\gamma\|u-v\|$$
$$\Rightarrow(1-\alpha-2\beta-\gamma)\|u-v\|\leq 0$$

$\Rightarrow\|u-v\|=0$, Therefore $u=v$.

Hence u is a unique fixed point of T.

Theorem 3.2: Let C be the non-empty closed subset of a Hilbert space H

Let $F,G: C\rightarrow C$ be the two self –mappings satisfying the following Condition

$$\|Fx-Gy\|\leq\alpha\frac{\|x-Fx\|+\|y-Gy\|+\|x-y\|}{1+\|x-Fx\|\|x-Gy\|\|y-Gy\|\|x-y\|}$$
$$+\beta\frac{\|x-Gy\|^2+\|y-Fx\|^2}{\|x-Gy\|+\|y-Fx\|}+\gamma\|x-y\|$$

For all $x,y\in C, x\neq y,\ 0\leq\alpha,\beta<\frac{1}{2}\ 0\leq\gamma$ and $3\alpha+2\beta+\gamma<1$.

Then F and G have a unique common fixed point in C.

Proof: Let C be a non-empty closed subset of Hilbert space H and $F,G: C\rightarrow C$ are the two self –mappings.

Let $x_0\in C$ is an arbitrary. We define a sequence $\{x_n\}$ in C as follows

$x_{2n+1}=Fx_{2n},\qquad x_{2n+2}=Gx_{2n+1},$

For $n=0,1,2,3,\dots\dots\dots\dots\dots$

If for some $n,\ x_n=x_{n+1}=x_{n+2}$

Then it is immediately follows x_n is a common fixed point of F and G.

Now we assume that there are no three consecutive terms equal in $\{x_n\}$.

We have

$$\|x_{2n+1}-x_{2n}\| = \|Fx_{2n}-Gx_{2n-1}\|$$

$$\leq \alpha \frac{\|x_{2n}-Fx_{2n}\| + \|x_{2n-1}-Gx_{2n-1}\| + \|x_{2n}-x_{2n-1}\|}{\|x_{2n}-Fx_{2n}\|\|x_{2n}-Gx_{2n-1}\|\|x_{2n-1}-Gx_{2n-1}\|\|x_{2n}-x_{2n-1}\|}$$

$$+\beta \frac{\|x_{2n}-Gx_{2n-1}\|^2 + \|x_{2n-1}-Fx_{2n}\|^2}{\|x_{2n}-Gx_{2n-1}\| + \|x_{2n-1}-Fx_{2n}\|} + \gamma\|x_{2n}-x_{2n-1}\|$$

$$= \alpha \frac{\|x_{2n}-x_{2n+1}\| + \|x_{2n-1}-x_{2n}\| + \|x_{2n}-x_{2n-1}\|}{1+\|x_{2n}-x_{2n+1}\|\|x_{2n}-x_{2n}\|\|x_{2n-1}-x_{2n}\|}$$

$$+\beta \frac{\|x_{2n}-x_{2n}\|^2 + \|x_{2n-1}-x_{2n+1}\|^2}{\|x_{2n}-x_{2n}\| + \|x_{2n-1}-x_{2n+1}\|} + \gamma\|x_{2n}-x_{2n-1}\|$$

$$\leq \alpha\{\|x_{2n}-x_{2n+1}\| + 2\|x_{2n-1}-x_{2n}\|\} + \beta\|x_{2n-1}-x_{2n+1}\| + \gamma\|x_{2n}-x_{2n-1}\|$$

$$(1-\alpha-\beta)\|x_{2n+1}-x_{2n}\| \leq (2\alpha+\beta+\gamma)\|x_{2n}-x_{2n-1}\|$$

$$\Rightarrow \|x_{2n+1}-x_{2n}\| \leq \left(\frac{2\alpha+\beta+\gamma}{1-\alpha-\beta}\right)\|x_{2n}-x_{2n-1}\|$$

$$\Rightarrow \|x_{2n+1}-x_{2n}\| \leq k\|x_{2n}-x_{2n-1}\|$$

Where $s = \left(\frac{2\alpha+\beta+\gamma}{1-\alpha-\beta}\right) < 1$ since $3\alpha+2\beta+\gamma<1$.

In general, $\|x_{n+1}-x_n\| \leq s\|x_n-x_{n-1}\| \leq s^2\|x_{n-1}-x_{n-2}\| \leq \dots\dots\dots \leq s^n\|x_1-x_0\|$

$i.e., \|x_{n+1}-x_n\| \leq s^n\|x_1-x_0\|$, For all $n \geq 1$.

Now we will prove that $\{x_n\}$ is a Cauchy sequence for this

$$\|x_n-x_{n+p}\| \leq \|x_n-x_{n+1}\| + \|x_{n+1}-x_{n+2}\| + \dots\dots\dots + \|x_{n+p-1}-x_{n+p}\|$$

$$\leq (s^n+s^{n+1}+s^{n+2}+\cdots\dots\dots+s^{n+p-1})\|x_1-x_0\|$$

$$= \left(\frac{s^n}{1-s}\right)\|x_1-x_0\|$$

$\Rightarrow \|x_n-x_{n+p}\| \to 0$, as $n\to\infty$, since $s<1$.

Therefore $\{x_n\}$ is a Cauchy sequence.

Since C is a closed subset of Hilbert space.

Therefore there exist an element $u \in C$ such that $\lim_{n\to\infty} x_n = u$.

Now we will prove that u is a common fixed point of F and G.

Firstly we will prove that u is a fixed point of F.

Let $Fu \neq u$.

Consider $\|u-Fu\| = \|u-x_{2n+2}+x_{2n+2}-Fu\|$

$$\leq \|u-x_{2n+2}\| + \|x_{2n+2}-Fu\|$$

$$= \|x_{2n+2}-Fu\|$$

$$= \|Fu-Gx_{2n+1}\|$$

$$\leq \alpha \frac{\|u-Fu\| + \|x_{2n+1}-Gx_{2n+1}\| + \|u-x_{2n+1}\|}{\|u-Fu\|\|u-Gx_{2n+1}\|\|x_{2n+1}-Gx_{2n+1}\|\|u-x_{2n+1}\|}$$

$$+\beta \frac{\|u-Gx_{2n+1}\|^2 + \|x_{2n+1}-Fu\|^2}{\|u-Gx_{2n+1}\| + \|x_{2n+1}-Fu\|} + \gamma\|u-x_{2n+1}\|$$

$$\textit{Solving and taking } n\to\infty \Rightarrow \|u-Fu\| \leq 0$$

$$\Rightarrow u = Fu$$

Hence u is a fixed point of F.

Similarly it can be proved that u is a fixed point of G.

Hence u is a common fixed point of F and G.

Uniqueness: It can be proved easily as Theorem 3.1

References

[1] Beg, I. and Shahzad, N. "Random fixed points of random multivalued operators on Polish spaces" Nonlinear Anal. 20 (1993) no. 7, 835-847.

[2] Beg, I. and Shahzad, N. "Random approximations and random fixed point theorems" J. Appl. Math. Stochastic Anal. 7 (1994). No 2, 145-150.

[3] Beg, I. and Shahzad, N. "Random fixed points of weakly inward operators in conical shells" J. Appl. Math, Stoch. Anal. 8(1995) 261-264.

[4] Bharucha –Reid, A.T. Fixed point theorems in probabilistic analysis, Bull. Amer. Math. Soc.82 (1976) 641-657.

[5] Choudhary, B.S. and Ray, M. "Convergence of an iteration leading to a solution of a random operator equation" J. Appl. Math. Stochastic Anal. 12 (1999). No 2,161-168.

[6] Choudhary, B.S. and Upadhyay, A. "An iteration leading to random solutions and fixed points of operators" Soochow J. Math.25 (1999). No 4,395-400.

[7] Choudhary, B.S. "A common unique fixed point theorem for two random operators in Hilbert spaces" I. J. M.M. S. 32 (2002) 177-182.

[8] Dhagat, V.B., Sharma, A. and Bhardwaj, R.K. "Fixed point theorem for random operators in Hilbert spaces" International Journal of Math. Analysis 2(2008) no. 12, 557-561.

[9] Kanan R, "construction of fixed point of a class of nonlinear mappings", J. Math. Appl. 41 (1973), P-430-438.

[10] Koparde, P.V. and Waghmode, D.B. "Kanan Type mappings in Hilbert spaces" Scientist of physical sciences, 3(1) (1991)45-50

[11] Shrivastava, Salu & Nair Smita, "A common fixed point theorem in Hilbert space" Acta ciencia indica, vol xxxII M, No.3, (2006)995-996.

[12] Sharma, A.K., Badshah, V.H. and Gupta, V. K.: Common fixed point theorems in Hilbert spaces, I.J.A.M.S.S., Vol.3 (1), (2014), 63-70.

[13] Sharma, A.K., Badshah, V.H. and Gupta, V. K.: Common fixed point theorems of a sequence of mappings in Hilbert spaces, Ultrascientist Phyl. Sci., (2012).